

SCALAR EXTENSIONS OF DERIVED CATEGORIES AND NON-FOURIER-MUKAI FUNCTORS

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ABSTRACT. Orlov’s famous representability theorem asserts that any fully faithful exact functor between the bounded derived categories of coherent sheaves on smooth projective varieties is a Fourier-Mukai functor. This result has been extended by Lunts and Orlov to include functors from perfect complexes to quasi-coherent complexes. In this paper we show that the latter extension is false without the full faithfulness hypothesis.

Our results are based on the properties of scalar extensions of derived categories, whose investigation was started by Pawel Sosna and the first author.

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1. INTRODUCTION

Unless otherwise specified, k is an algebraically closed base field of characteristic zero. Orlov’s famous representability theorem [30, Thm 2.2] asserts that any fully faithful exact functor between the bounded derived categories of coherent sheaves on smooth projective varieties over k is a Fourier-Mukai functor. It is still unknown if

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the full faithfulness hypothesis is necessary in this theorem, although some positive results were obtained by the first author in [35].

A number of extensions and variants of Orlov's theorem are known. See e.g. [2, 6, 7, 8, 10, 20, 27]. For an excellent survey on the current state of knowledge see [9]. In particular, Lunts and Orlov proved the following natural extension of Orlov's theorem to quasi-coherent sheaves:

Proposition A. [27, Corollary 9.13(2)] *Let X/k be a projective scheme such that \mathcal{O}_X has no zero dimensional torsion and let Y be a quasi-compact separated scheme. Then every fully faithful exact functor $\Psi : \text{Perf}(X) \rightarrow D(\text{Qcoh}(Y))$ is isomorphic to the restriction of a Fourier-Mukai functor associated to an object in $D(\text{Qcoh}(X \times Y))$.*

One of the main results of this paper is that this extension is false if we drop the condition that Ψ is fully faithful, even in the case that X, Y are smooth and projective (see Theorem 9.1 below). Our arguments are based on the properties of scalar extensions of derived categories, which we will outline below. We will get back to Proposition A at the end of the introduction.

If \mathfrak{a} is a k -linear category and B is a k -algebra, we denote by \mathfrak{a}_B the category of B -objects in \mathfrak{a} , i.e. pairs (M, ρ) where $M \in \text{Ob}(\mathfrak{a})$ and $\rho : B \rightarrow \mathfrak{a}(M, M)$ is a k -algebra morphism. If \mathcal{C} is abelian then so is \mathcal{C}_B , but if \mathcal{T} is triangulated there is no reason for this to be the case for \mathcal{T}_B as well.

While investigating generalizations of Orlov's theorem [36] the first author studied the obvious forgetful functor

$$F : D^b(\mathcal{C}_B) \rightarrow D^b(\mathcal{C})_B$$

for $B/k = L/k$ a field extension. She proved an essential surjectivity result for $\text{trdeg } L/k \leq 2$ but it appeared difficult to go beyond that. Indeed, in the present paper we will show that F is generally *not* essentially surjective when $\text{trdeg } L/k = 3$. To put this in context, we start with a positive result which is naturally proved using A_∞ -techniques:

Proposition B. (See Propositions 10.1.1, 10.1.2, 10.1.3 below.) *Assume that \mathcal{C} is a Grothendieck category.*

- *If B/k has Hochschild dimension ≤ 2 , F is essentially surjective.*
- *If B/k has Hochschild dimension ≤ 1 , F is in addition full.*
- *If B/k has Hochschild dimension 0, F is an equivalence of categories.*

Recall that, for a finitely generated field extension L/k , the Hochschild dimension is equal to the transcendence degree. Proposition B represents a strengthening of the results in [21]. The case of Hochschild dimension 0 generalises results by Sosna [41].

However, our next result shows that one cannot hope to substantially improve Proposition B:

Theorem C. (See Theorem 8.1 below.) *Let X/k be a smooth connected projective variety which is not a point, a projective line or an elliptic curve. Then there exists a finitely generated field extension L/k of transcendence degree 3 together with an object $Z \in D^b(\text{Qcoh}(X))_L$ which is not in the essential image of F .*

The proof of this theorem will depend on a similar result for representations of wild quivers, which we will prove first (see Proposition 7.1 below).

As the reader may notice, Theorem C leaves out the case where X is a curve of genus ≤ 1 . The key point is that in this case the moduli space of indecomposable objects has dimension ≤ 1 . We capture this in the concept of “essential dimension”, which is roughly speaking the minimal number of parameters required to define any family of indecomposable objects (see Definition 6.2.1 below for a precise definition). From this theory it follows that if X is a curve of genus ≤ 1 and $\mathcal{C} = \mathrm{Qcoh}(X)$, for any field extension L/k the essential image of F contains all objects in $D^b(\mathrm{Qcoh}(X))_L$ whose cohomology lies in $\mathrm{coh}(X_L) \subset \mathrm{Qcoh}(X_L) \cong \mathrm{Qcoh}(X)_L$ (see Remark 6.2.2 and Theorem 6.2.3 below). We suspect that F is in fact essentially surjective, but we have not proved it.

Now we come back to Proposition A. A counterexample to this proposition, when dropping the full faithfulness hypothesis, may be obtained using the following result:

Theorem D. *(See Theorem 9.1 below) Let X, Y be connected smooth projective schemes. Let $i_\eta : \eta \rightarrow X$ be the generic point of X and let $L = k(\eta)$ be the function field of X . Assume that $D^b(\mathrm{Qcoh}(Y))_L$ contains an object Z which is not in the essential image of $D^b(\mathrm{Qcoh}(Y))_L$ (for example as in Theorem C). Define Ψ as the composition*

$$\mathrm{Perf}(X) \xrightarrow{i_\eta^*} D(L) \xrightarrow{L \mapsto Z} D(\mathrm{Qcoh}(Y)) .$$

Then Ψ is not the restriction of a Fourier-Mukai functor.

We start with a few technical sections that will provide tools for the proofs of the main results. The impatient reader may wish to proceed directly to §6 at first reading.

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3. MODULI SPACES OF REPRESENTATIONS OF ALGEBRAS

Moduli space of representations for algebras may be constructed in several different ways [22, 32]. We remind the reader of a construction which is based on the properties of the Formanek center and which will be used in the proof Lemma 3.2.1 and Proposition 3.5.2, which are the main results of this section. We will use our standing characteristic zero hypothesis to simplify the discussion.

3.1. The representation functor. For a reference on this subject, the reader can consult [32]. An Azumaya algebra is a matrix algebra for the étale topology. Let A be a k -algebra and let $n > 0$. Consider the functor $\text{Az}_{n,A}$ from commutative k -algebras to the category of sets defined as follows: if R is a commutative k -algebra, $\text{Az}_{n,A}(R)$ is the set of equivalence classes of maps of k -algebras $\rho : A \rightarrow B$, where B is an Azumaya algebra of rank n^2 over R satisfying $\rho(A)R = B$. Two maps $\rho : A \rightarrow B$, $\rho' : A \rightarrow B'$ are considered equivalent if there exists an isomorphism of R -algebras $\xi : B \rightarrow B'$ such that $\xi\rho = \xi'$. The functor $\text{Az}_{n,A}$ is a sheaf for the Zariski topology, and hence extends canonically to a functor from k -schemes to the category of sets.

The functor $\text{Az}_{n,A}$ is representable in the category of k -schemes (see [32, Ch IV, Thm 1.8 and Ch VIII, Thm 2.2]). The representing scheme may be constructed as a “Formanek center”, constructed as follows. Let Λ be a k -algebra. The identities of $n \times n$ matrices with coefficients in Λ are defined as

$$I = \{f(x_s) \in \Lambda\{X_s\} \mid f(r_s) = 0 \text{ for all } r_s \in (R)_n\},$$

where $(R)_n$ denotes the algebra of $n \times n$ matrices with coefficients in R , and R is a generic commutative algebra in the variety of commutative algebras (see [32, Definition 4.3]). If a ring C satisfies the identities of $n \times n$ matrices, the Formanek center $F(C)$ of C is defined as the subring of C obtained by evaluating all central polynomials of $n \times n$ -matrices without constant term. A central polynomial for $n \times n$ matrices is a polynomial in non-commuting variables that is non-constant, but yields a scalar matrix whenever it is evaluated at $n \times n$ matrices.

By definition, $F(C) \subset Z(C)$. Since the field is of characteristic zero, the usual polarization argument [33] shows that we may compute the Formanek center by evaluating central polynomials which are homogeneous of degree one in every variable. From this it easily follows that it is an ideal.¹ An algebra satisfying the identities of $n \times n$ -matrices is Azumaya of rank n^2 over its center if and only if the Formanek center is equal to the ordinary center (this follows easily from [32, Ch VIII, Th 2.1(6)], using the characteristic zero hypothesis again).

Let A_n be the quotient of A by the identities of $n \times n$ -matrices, and let $F_{n,A} = F(A_n)$ be the Formanek center of A_n . Put $F_{n,A}^e = F^e(A_n) \stackrel{\text{def}}{=} k + F(A_n) \subset A_n$ and $\tilde{U}_{n,A} = \text{Spec } F_{n,A}^e$. Let $\tilde{\mathcal{A}}_n$ be the sheaf of algebras on $\tilde{U}_{n,A}$ associated to A_n and put $U_{n,A} \stackrel{\text{def}}{=} \tilde{U}_{n,A} - V(F_{n,A})$. Finally, let \mathcal{A}_n be the restriction of $\tilde{\mathcal{A}}_n$ to $U_{n,A}$. Then \mathcal{A}_n is a sheaf of Azumaya algebra of rank n^2 with center equal to $\mathcal{O}_{U_{n,A}}$ (presumably this depends on the characteristic zero hypothesis, see [32, Ch VIII, Cor. 2.3] for a result valid in any characteristic). Note that $U_{n,A}$ has an affine covering by schemes of the form $U_{n,A,f} = \text{Spec}(F_{n,A}^e)_f$, where f runs through the elements of $F_{n,A}$. The global sections of \mathcal{A}_n restricted to $U_{n,A,f}$ are given by $(A_n)_f$.

It follows from [32, Ch VIII, Thm 2.2] that $U_{n,A}$ is isomorphic to a differently constructed scheme which represents $\text{Az}_{n,A}$. For further reference we give a description of the bijection between $U_{n,A}(R)$ and $\text{Az}_{n,A}(R)$ as given in the proof of [32, Ch VIII, Thm 2.2]. Assume $\rho : A \rightarrow B$ represents an element of $\text{Az}_{n,A}(R)$. The map ρ descends to a map $\rho_n : A_n \rightarrow B$, and hence to a map $\rho_{n,f} : (A_n)_f \rightarrow B_{\rho_n(f)}$ for $f \in F_{n,A}$. We obtain an induced map $\rho_{n,f} : F^e((A_n)_f) \rightarrow F^e(B_{\rho_n(f)})$. Now $B_{\rho_n(f)}$ is an Azumaya algebra and hence $F^e(B_{\rho_n(f)})$ is equal to its center $R_{\rho_n(f)}$.

¹We do not know if this is true in finite characteristic.

It is easy to see that the maps $(F_{n,A}^e)_f \rightarrow R_{\rho_n}(f)$ may be glued to a scheme map $\rho_n : \text{Spec } R \rightarrow \text{Spec } F_{n,A}^e - V(F_{n,A}) = U_{n,A}$.

Conversely, if we start from a scheme map $\rho_n : \text{Spec } R \rightarrow U_{n,A}$, then we put $B = \rho_n^*(\mathcal{A}_n)$ (where here and below we usually identify quasi-coherent sheaves on affine schemes with their global sections). Since the elements of A restrict to sections of \mathcal{A}_n , we obtain a corresponding map $\rho : A \rightarrow B$. The required condition $\rho(A)R = B$ is easily checked.

3.2. The main lemma. Let the notation be as in the previous section. For a not necessarily closed point $i_x : x \rightarrow U_{n,A}$ we say that x is split if $i_x^*(\mathcal{A}_n)$ is split as a central simple algebra, i.e. if it is isomorphic to $M_n(k(x))$ as a $k(x)$ -algebra. In this case, V_x is defined to be the corresponding irreducible $A_{k(x)} \stackrel{\text{def}}{=} k(x) \otimes_k A$ -representation of dimension n over $k(x)$.

Lemma 3.2.1. *Assume x is split. We have $\text{End}_A(V_x) = k(x)$.*

The point of the lemma is that the endomorphisms are only assumed to be A -linear, not $A_{k(x)}$ -linear.

Proof. Let $O(x)$ be the image of $F_{n,A}^e$ in $k(x)$. Then $k(x)$ is the field of fractions of $O(x)$ and we have

$$\begin{aligned} \text{End}_A(V_x) &= \text{End}_{A_n}(V_x) \\ &= \text{End}_{O(x) \otimes_{F_{n,A}^e} A_n}(V_x) \\ &= \text{End}_{k(x) \otimes_{F_{n,A}^e} A_n}(V_x) \\ &= \text{End}_{A_{k(x)}}(V_x) \\ &= k(x). \end{aligned}$$

In the second equality we use that $A_n \rightarrow O(x) \otimes_{F_{n,A}^e} A_n$ is surjective. In the third equality we use that $O(x) \otimes_{F_{n,A}^e} A_n \rightarrow k(x) \otimes_{F_{n,A}^e} A_n$ is an epimorphism of rings and the fact that V_x is a $k(x) \otimes_{F_{n,A}^e} A_n$ -module. For the fourth equality we use that $A_{k(x)} \rightarrow k(x) \otimes_{F_{n,A}^e} A_n$ is surjective. \square

Example 3.2.2. Here is an example where one can check the conclusion of Lemma 3.2.1 directly. Let Q be the quiver with one vertex and three loops and $A = kQ = k\langle X, Y, Z \rangle$. Then it is easy to see that $U_{1,A} \cong \mathbb{A}^3$. Let V_η be the representation corresponding to the generic point η of \mathbb{A}^3 . It is defined over the field $L = k(\eta) = k(x, y, z)$ and has the form

One easily checks that $\text{End}_{k\langle X, Y, Z \rangle}(V_\eta) = L$.

3.3. The split representation functor. In order to use Lemma 3.2.1, we must be able to show that $i_x^*(\mathcal{A}_n)$ is split. We discuss this next. For a k -scheme X , let $\mathcal{M}_{n,A}(X)$ be the collection of equivalence classes of quasi-coherent sheaves of left $A \otimes_k \mathcal{O}_X$ -modules V on X which are vector bundles or rank n over X such that for every point $x \in X$ we have that $i_x^*(V)$ is simple. We consider V and W to be equivalent if there exists an invertible \mathcal{O}_X -module I such that $W = V \otimes_X I$.

Lemma 3.3.1. *Assume that $\mathcal{M}_{n,A}$ is representable in the category of k -schemes. Then \mathcal{A}_n is split and $\mathcal{M}_{n,A}$ is represented by $U_{n,A}$.*

Proof. Let $M_{n,A}$ be the representing scheme for $\mathcal{M}_{n,A}$, and let $\mathcal{V}_{n,A}$ be the universal bundle on $M_{n,A}$ (determined up to tensoring with a line bundle). We have a natural transformation

$$\phi : \mathcal{M}_{n,A} \rightarrow \text{Az}_{n,A}$$

sending V to $\text{End}_X(V)$. This yields a morphism between the representing schemes

$$(3.1) \quad \phi : M_{n,A} \rightarrow U_{n,A}$$

such that

$$(3.2) \quad \phi^* \mathcal{A}_n = \text{End}_{M_{n,A}}(\mathcal{V}_{n,A}).$$

Clearly, $\phi(X)$ is injective for any X , and surjective up to tale coverings. This means that ϕ is actually an isomorphism and hence \mathcal{A}_n is split by (3.2). \square

3.4. Partitioning by ranks. Let $l = ke_1 + \dots + ke_m$ be the semi-simple k -algebra determined by $e_i e_j = \delta_{ij} e_i$, $\sum_i e_i = 1$. Let A be an l -algebra. By this we mean that there is given a k -algebra morphism $l \rightarrow A$. We denote the images of the $(e_i)_i$ in A also by $(e_i)_i$. Fix positive natural numbers $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $|\alpha| \stackrel{\text{def}}{=} \sum_i \alpha_i = n$. We let $\text{Az}_{\alpha,A}(R)$ be the subset of $\text{Az}_{n,A}(R)$ consisting of equivalence classes $\rho : A \rightarrow B$ such that $\text{rk}_R \rho(e_i) B \rho(e_i) = \alpha_i^2$ for all i . It is easy to see that $\text{Az}_{\alpha,A}$ is an open subfunctor of $\text{Az}_{n,A}$, and furthermore $\text{Az}_{n,A} = \coprod_{\alpha} \text{Az}_{\alpha,A}$. We obtain a corresponding decomposition $U_{n,A} = \coprod_{\alpha} U_{\alpha,A}$ for the representing spaces. The restriction of \mathcal{A}_n to $U_{\alpha,A}$ will be denoted by \mathcal{A}_{α} .

In a similar way we may define functors $\mathcal{M}_{\alpha,A} \subset \mathcal{M}_{n,A}$, where we now require that for $V \in \mathcal{M}_{\alpha,A}(X)$ the rank of $e_i V$ is equal to α_i . We will say that V has *dimension vector* α . We have the following obvious generalization of Lemma 3.3.1:

Lemma 3.4.1. *Assume that $\mathcal{M}_{\alpha,A}$ is representable in the category of k -schemes. Then \mathcal{A}_{α} is split and $\mathcal{M}_{\alpha,A}$ is represented by $U_{\alpha,A}$.*

3.5. Stability conditions. For $\lambda, \mu \in \mathbb{Z}^m$ write $\lambda \cdot \mu = \sum_{i=1}^m \lambda_i \mu_i$. Let $\alpha \in \mathbb{N}^m$, and choose $\lambda \in \mathbb{Z}^m$ such that $\lambda \cdot \alpha = 0$.

If K/k is a field extension, then $V \in \mathcal{M}_{\alpha,A}(K)$ is λ -(semi-)stable [22] if for any proper A_K -subrepresentation $0 \neq W \subsetneq V$ with dimension vector β we have

$$\lambda \cdot \beta(\geq) > \lambda \cdot \alpha.$$

We say that $V \in \mathcal{M}_{\alpha,A}(X)$ is λ -(semi-)stable if for any $i : \text{Spec } K \rightarrow X$ with K/k a field extension it is true that $i^*(V)$ is λ -(semi-)stable. We denote the corresponding subfunctor of $\mathcal{M}_{\alpha,A}$ by $\mathcal{M}_{\alpha,\lambda,A}$. Recall

Theorem 3.5.1. *Assume that A is finitely generated over k and that α is indivisible. Then $\mathcal{M}_{\alpha,\lambda,A}$ is representable by a scheme of finite type over k .*

This is [22, Prop 5.3]. The given reference assumes A to be finite dimensional, but the proof carries over completely in the case where A is finitely generated over k .

We denote the representing scheme by $M_{\alpha,\lambda,A}$, and the corresponding universal A -representation by $\mathcal{V}_{\alpha,\lambda,A}$. We will prove the following generalization of Lemma 3.2.1. The point is again that we take endomorphisms over A , and not over $A_{k(x)}$.

²We assume $0 \in \mathbb{N}$.

Proposition 3.5.2. *Let A be finitely generated, and let α be indivisible. For $i_x : x \rightarrow M_{\alpha,\lambda,A}$ put $V_x = i_x^*(\mathcal{V}_{\alpha,\lambda,A})$. Then*

$$(3.3) \quad \text{End}_A(V_x) = k(x).$$

The proof of Proposition 3.5.2 uses the representation theory of quivers. See [5] for a reference. Let $Q = (Q_0, Q_1, h, t)$ be a finite quiver with vertices Q_0 and arrows Q_1 . The maps $h, t : Q_1 \rightarrow Q_0$ associate an arrow with its head and tail. If R is a k -algebra, we let RQ be the path algebra of Q with coefficients in R . Any finitely generated l -algebra in the sense of §3.4 is a quotient of a suitable path algebra kQ with l corresponding to $\bigoplus_{i \in Q_0} ke_i$, where e_i is the length zero path in Q associated to the vertex i . It is easy to see that it is sufficient to prove Proposition 3.5.2 in the case $A = kQ$. Therefore we specialize to that case, and we will replace A in the notation by kQ .

For a kQ -representation V we write $V_i = e_i V$. Let $P_i = kQe_i$ be the standard projective representation corresponding to $i \in Q_0$. Recall the following:

Proposition 3.5.3. *(Green) Every projective kQ -module P is of the form $\bigoplus_{i \in Q_0} W_i \otimes_k P_i$, where W_i is a k -vector space. The W_i are uniquely determined by P .*

Proof. The fact that all projectives are of the indicated form is [15, Cor. 5.5]. Then W_i can be recovered from P as $W_i = S_i \otimes_{kQ} P$, where S_i is the standard simple corresponding to vertex i . \square

Below we will need a definition which is dual to the concept of dimension vector $\underline{\dim} V$ of V . Assume that $V \in \text{Mod}(kQ)$ is finitely presented. Then V has a projective resolution

$$0 \rightarrow \bigoplus_i P_i^{\oplus b_i} \rightarrow \bigoplus_i P_i^{\oplus a_i} \rightarrow V \rightarrow 0$$

with P_i the projective kQ -representation associated to vertex i . We put

$$(\overline{\dim} V)_i = a_i - b_i.$$

It follows from Proposition 3.5.3 that this is a well defined element of \mathbb{Z}^{Q_0} .

For W a finitely presented and V a finite dimensional kQ -representation, one has

$$\overline{\dim} W \cdot \underline{\dim} V = \dim \text{Hom}_{kQ}(W, V) - \dim \text{Ext}_{kQ}^1(W, V).$$

If $\text{Hom}_{kQ}(W, V) = \text{Ext}_{kQ}^1(W, V) = 0$ then we write $W \perp V$. Let W^\perp be the category of all V such that $W \perp V$. Recall the following result

Theorem 3.5.4. [11, 12][40, Cor 1.1] *Assume that $V \in \text{Mod}(kQ)$ is finite-dimensional. Then V is λ -semi-stable if and only there is a finitely presented $0 \neq W \in \text{Mod}(kQ)$ such that $W \perp V$, and such that $\overline{\dim} W$ is a strictly negative multiple of λ .*

For $\lambda \cdot \alpha = 0$, and for W such that $\overline{\dim} W = -n\lambda$, $n > 0$ we define a subfunctor $\mathcal{M}_{\alpha,W,Q}$ of $\mathcal{M}_{\alpha,\lambda,Q}$ consisting of those representations V with $\underline{\dim} V = \alpha$ such that $W \perp V$. If α is indivisible, then this subfunctor is representable by an open subset $M_{\alpha,W,Q}$ of $M_{\alpha,\lambda,Q}$. The above theorem may be rephrased as saying that $(M_{\alpha,W,Q})_W$ is an open covering of $M_{\alpha,\lambda,Q}$. We let $\mathcal{V}_{\alpha,W,Q}$ be the restriction of $\mathcal{V}_{\alpha,\lambda,Q}$ to $M_{\alpha,W,Q}$.

Let

$$0 \rightarrow P \xrightarrow{\delta} Q \rightarrow W \rightarrow 0$$

be a minimal projective resolution of W and let $(kQ)_\delta$ be the corresponding universal localisation.³

Then $V \in W^\perp$ if and only if $\text{Hom}_{kQ}(\delta, V)$ is invertible. In other words, $V \in W^\perp$ if and only if the kQ -action on V extends to a $(kQ)_\delta$ -action. Thus we obtain

$$(3.4) \quad W^\perp \cong \text{Mod}((kQ)_\delta).$$

Note further

Lemma 3.5.5. [1, Lemma 3.1] *Let W be a finitely presented kQ -representation and let $V \in W^\perp \cap \text{Mod}(KQ)$ be finite dimensional over K , where K/k is a field extension. Then V is simple in $W^\perp \cap \text{Mod}(KQ)$ if and only if V is $-\overline{\dim} W$ -stable.*

Proof. The blanket hypothesis in [1] that Q should not have any oriented cycles is not necessary for this particular lemma. For the benefit of the reader we recall the proof of the implication $\text{simple} \Rightarrow \text{stable}$, as it is instructive. Assume that V is simple in $W^\perp \cap \text{Mod}(KQ)$, and let $0 \neq V' \subsetneq V$ be a KQ -subrepresentation. Since $\text{Hom}_{kQ}(W, V') = 0$, we have $\overline{\dim} W \cdot \underline{\dim} V' \leq 0$. If $\overline{\dim} W \cdot \underline{\dim} V' = 0$ then we also have $\text{Ext}_{kQ}^1(W, V') = 0$ and hence $W \perp V'$. Since V is simple this is a contradiction. It follows $\dim \text{Ext}_{kQ}^1(W, V') > 0$, and so $\overline{\dim} W \cdot \underline{\dim} V' < 0$. \square

From this we easily obtain an isomorphism of functors

$$\mathcal{M}_{\alpha, (kQ)_\delta} \cong \mathcal{M}_{\alpha, W, Q}$$

and hence, by Lemma 3.4.1, it follows that $\mathcal{A}_{\alpha, (kQ)_\delta}$ is split and that there is an isomorphism

$$U_{\alpha, (kQ)_\delta} \cong M_{\alpha, W, Q}.$$

Proof of Proposition 3.5.2. There exists some W such that $x \in M_{\alpha, W, Q}$. Now let the notation be as above. Then by (3.4)

$$\text{End}_{kQ}(V_x) = \text{End}_{(kQ)_\delta}(V_x).$$

It now suffices to invoke Lemma 3.2.1. \square

Example 3.5.6. Let Q be the generalised Kronecker quiver with 4 arrows.

$$\begin{array}{ccc} & \xrightarrow{T} & \\ \textcircled{1} & \xrightarrow{X} & \textcircled{2} \\ & \xrightarrow{Y} & \\ & \xrightarrow{Z} & \end{array}$$

Let $\alpha = (1, 1)$ and $\lambda = (-1, 1)$. A representation of dimension vector α is λ -stable if not all arrows are zero, and two such representations are isomorphic if one is obtained from the other by multiplying all arrows by the same scalar. This corresponds to a point in \mathbb{P}^3 . From there one easily shows that $M_{\alpha, \lambda, A} = \mathbb{P}^3$. Let V_η be the representation corresponding to the generic point η of \mathbb{P}^3 . It is defined over the field $L = k(\eta) = k(x, y, z)$ and has the form

$$\begin{array}{ccc} & \xrightarrow{1} & \\ L & \xrightarrow{x} & L \\ & \xrightarrow{y} & \\ & \xrightarrow{z} & \end{array}$$

One easily checks that $\text{End}_A(V_\eta) = L$.

³If A is a ring and $\delta : P \rightarrow Q$ is map between finitely generated projective left A -modules then $A \rightarrow A_\delta$ is universal for the ring extensions $A \rightarrow B$ such that $B \otimes_A \delta$ is an isomorphism. $A \rightarrow A_\delta$ is an epimorphism in the category of rings and a left A -module M has a (necessarily unique) A_δ -action provided the functor $\text{Hom}_A(-, M)$ transforms δ into an invertible morphism.

If P_1, P_2 denote the projective Q -representations corresponding to the vertices 1, 2 and $W = \text{coker}(P_2 \xrightarrow{T} P_1)$ then $V_\eta \in W^\perp$. The universal localisation of kQ at T is obtained by adjoining to Q an inverse arrow T^{-1} from 2 to 1. One checks that $(kQ)_T$ is Morita equivalent to $k\langle X, Y, Z \rangle$, and this example reduces in fact to Example 3.2.2.

It is difficult to say when $M_{\alpha, \lambda, Q}$ is non-empty, but Proposition 3.5.9 below will be sufficient for our purposes. For $\alpha, \beta \in \mathbb{Z}^{Q_0}$ write

$$\langle \alpha, \beta \rangle = \sum_i \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

If W, V are finite dimensional kQ -representations then

$$\langle \underline{\dim} W, \underline{\dim} V \rangle = \dim \text{Hom}_{kQ}(W, V) - \dim \text{Ext}_{kQ}^1(W, V).$$

Put

$$(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$$

and let $e_i \in \mathbb{N}^{Q_0}$ be such that $(e_i)_j = \delta_{ij}$. The fundamental region [19] is defined as

$$F(Q) = \{ \alpha \in \mathbb{N}^{Q_0} \mid \forall i : (e_i, \alpha) \leq 0, \alpha \text{ has connected support} \}.$$

Recall

Lemma 3.5.7. *$F(Q)$ is empty for a Dynkin quiver and is spanned by a single vector δ satisfying $(\delta, \delta) = 0$ in the case that Q is extended Dynkin. If Q contains a component with more than one vertex which is not Dynkin or extended Dynkin then $F(Q)$ contains indivisible α such that (α, α) is an arbitrarily large negative number.*

Proof. The first two cases are well known. So supposed that Q is connected and not Dynkin or extended Dynkin and $|Q_0| > 1$. By [19, Lemma 1.2] there exists $\alpha \in \mathbb{N}^{Q_0}$ such that all $\alpha_i > 0$ and $(e_i, \alpha) < 0$ for each i . In other words the cone

$$C(Q) = \{ \beta \in \mathbb{R}^{Q_0} \mid \forall i : \beta_i \geq 0, (e_i, \beta) \leq 0 \}.$$

had non empty interior and is of dimension $|Q_0| > 1$. So $\text{int } C(Q) \cap \mathbb{Z}^{Q_0}$ contains infinitely many indivisible elements (for example take the minimal elements in smaller and smaller subcones which are disjoint except for 0).

Now if $\beta \in \text{int } C(Q) \cap \mathbb{Z}^{Q_0}$ then $(\beta, \beta) = \sum_i \beta_i (\beta, e_i) < -\sum_i \beta_i$. So for any $N > 0$ the set

$$\{ \beta \in \text{int } C(Q) \cap \mathbb{Z}^{Q_0} \mid (\beta, \beta) > -N \}$$

is finite. This shows that $(\beta, \beta) \rightarrow -\infty$. \square

Remark 3.5.8. It follows that if Q is a connected “wild” quiver (i.e. not Dynkin or extended Dynkin) which is not the two loop quiver then $F(Q)$ contains an indivisible vector α such that $(\alpha, \alpha) \leq -4$. This fact will be used below.

Proposition 3.5.9. [19, 22, 39] *Let α be an indivisible dimension vector in $F(Q)$. Then there exists some λ satisfying $\lambda \cdot \alpha = 0$ such that $M_{\alpha, \lambda, Q}$ is non-empty. In that case $M_{\alpha, \lambda, Q}$ has dimension $-(1/2)(\alpha, \alpha) + 1$.*

Proof. Since $\alpha \in F(Q)$, the generic Q -representation with dimension vector α is a Schur representation (i.e. it has only trivial endomorphisms) [19]. Therefore it is stable for suitable λ by [38, Theorem 6.1]. The dimension maybe computed using the standard fact that $\dim M_{\alpha, Q} = \dim \text{Ext}_{kQ}^1(V, V) = -\langle \underline{\dim} V, \underline{\dim} V \rangle + 1$ for V generic. \square

4. MODULI SPACES OF VECTOR BUNDLES ON CURVES

In this section we prove an analogue of Proposition 3.5.2 for vector bundles on curves. Below X is a smooth projective curve over k of genus g . The theory of moduli spaces of vector bundles on curves is well known, so we will not repeat it here (see e.g. [29]).

Given r, d such that $\gcd(r, d) = 1$, the functor $\mathcal{M}_{r,d}$ of families of stable vector bundles of rank r and degree d on X has a fine moduli space $M_{r,d}$ such that

$$\dim M_{r,d} = 1 + r^2(g - 1).$$

Let $\mathcal{V}_{r,d}$ be the universal bundle on $M_{r,d}$. We will prove the following analogue of Proposition 3.5.2

Proposition 4.1. *Let $x \in M_{r,d}$ and put $V_x = i_x^*(\mathcal{V}_{r,d})$. Let $p : X_{k(x)} \rightarrow X$ be the map obtained by base extension from the structure map $\mathrm{Spec} k(x) \rightarrow \mathrm{Spec} k$. Then*

$$\mathrm{End}_X(p_* V_x) = k(x).$$

To prove Proposition 4.1 we will use the following analogue of Theorem 3.5.4, which is a fundamental result by Faltings:

Theorem 4.2. [13] *Let X be a smooth projective curve. A bundle \mathcal{E} on X is semi-stable if there exists a non-zero bundle \mathcal{F} such $\mathcal{F} \perp \mathcal{E}$.*

As before $\mathcal{F} \perp \mathcal{E}$ if $\mathrm{Hom}_X(\mathcal{F}, \mathcal{E}) = 0$, $\mathrm{Ext}_X^1(\mathcal{F}, \mathcal{E}) = 0$. Given $\mathcal{F} \in \mathrm{coh}(X)$, we define as in the quiver case a subfunctor $\mathcal{M}_{r,d,\mathcal{F}}$ of $\mathcal{M}_{r,d}$ consisting of those families in $\mathcal{M}_{r,d}$ that are right orthogonal to \mathcal{F} . This subfunctor is representable by an open subset of $M_{r,d}$, which we denote by $M_{r,d,\mathcal{F}}$.

Let $\mathcal{F} \in \mathrm{coh}(X)$ be such that $\mathrm{Supp} \mathcal{F} = X$. Put

$$\mathcal{F}^\perp = \{\mathcal{E} \in \mathrm{Qcoh}(X) \mid \mathrm{Hom}_X(\mathcal{F}, \mathcal{E}) = \mathrm{Ext}_X^1(\mathcal{F}, \mathcal{E}) = 0\}.$$

It is easy to see that \mathcal{F}^\perp is an abelian subcategory of $\mathrm{Qcoh}(X)$ closed under direct sums. So it is in particular a Grothendieck category. We will now use some results by Aidan Schofield, which are unfortunately not officially published. Proofs can be found in [37].

Proposition 4.3. [37] *The inclusion $\mathcal{F}^\perp \subset \mathrm{Qcoh}(X)$ has a left adjoint.*

Denote the left adjoint to $\mathcal{F}^\perp \rightarrow \mathrm{Qcoh}(X)$ by L . Let $p \in X$. There exists an epimorphism $\phi : \mathcal{F} \rightarrow \mathcal{O}_p$. Put $\mathcal{F}' = \ker \phi$, $\mathcal{P} = L(\mathcal{F}')$.

Proposition 4.4. [37] *The object \mathcal{P} is a small projective generator for the category \mathcal{F}^\perp . If $\mathcal{E} \in \mathcal{F}^\perp$ then $\mathrm{Hom}_X(\mathcal{P}, \mathcal{E})$ is finite dimensional if and only if \mathcal{E} is coherent.*

Put $A = \mathrm{End}_X(\mathcal{P})$. It follows that there is an equivalence of categories

$$(4.1) \quad \mathcal{F}^\perp \rightarrow \mathrm{Mod}(A) : \mathcal{E} \mapsto \mathrm{Hom}_X(\mathcal{P}, \mathcal{E})$$

which is an analogue to (3.4). With a similar argument as in the discussion thereafter, we obtain an isomorphism of functors

$$\mathcal{M}_{r,A} \cong \mathcal{M}_{r,d,\mathcal{F}}$$

and hence by Lemma 3.3.1 it follows that $\mathcal{A}_{r,A}$ is split and there is an isomorphism

$$U_{r,A} \cong M_{r,d,\mathcal{F}}.$$

Proof of Proposition 4.1. There exists some \mathcal{F} such that $x \in M_{r,d,\mathcal{F}}$. Now let the notation be as above. Then by (4.1)

$$\mathrm{End}_X(p_*V_x) = \mathrm{End}_A(V_x).$$

It now suffices to invoke Lemma 3.2.1. \square

5. HOMOLOGICAL IDENTITIES

We recall the basic notions regarding Hochschild cohomology. We state the definitions for graded algebras since this is the generality which we will need later. For a comprehensive introduction, see [42].

Let B be a graded k -algebra and let M be a graded B - B -bimodule. Construct the (graded) *Hochschild complex* as

$$C^i(B, M) = \mathrm{Hom}_{\mathrm{gr}(k)}(B^{\otimes i}, M) = \bigoplus_j \mathrm{Hom}_k(B^{\otimes i}, M)_j$$

where $\mathrm{Hom}_k(B^{\otimes i}, M)_j$ represents the set of k -multilinear maps $B^{\otimes i} \rightarrow M$ of degree j . The differential is given by

$$\begin{aligned} d_{\mathrm{Hoch}}(f)(r_0, \dots, r_i) &= r_0 f(r_1, \dots, r_i) - f(r_0 r_1, \dots, r_i) + \dots + (-1)^{i-1} f(r_0, \dots, r_{i-1} r_i) \\ &\quad + (-1)^i f(r_0, \dots, r_{i-1}) r_i. \end{aligned}$$

The Hochschild cohomology is defined as

$$\mathrm{HH}^i(B, M) = H^i C^*(B, M)$$

and has a decomposition $\mathrm{HH}^i(B, M) = \bigoplus_j \mathrm{HH}^i(B, M)_j$, where the elements of $\mathrm{HH}^i(B, M)_j$ are represented by the cocycles of degree j with i arguments.

The *Hochschild dimension* of a k -algebra B is the projective dimension of B as an $B \otimes_k B^{\mathrm{op}}$ -module. Equivalently, it is the maximum d such that there exists an B - B -bimodule M with $\mathrm{HH}^d(B, M) \neq 0$. If B is a finitely generated field extension of k , then [31] the Hochschild dimension of B equals its transcendence degree over k .

In this section we use Hochschild cohomology to compute Ext-groups in base extended categories. We first recall the following “change of rings” result:

Proposition 5.1. *Let \mathcal{C} be a k -linear Grothendieck category and let B be a k -algebra. Then for $M, N \in D(\mathcal{C}_B)$ we have*

$$(5.1) \quad \mathrm{RHom}_{B \otimes_k B^{\mathrm{op}}}(B, \mathrm{RHom}_{\mathcal{C}}(M, N)) = \mathrm{RHom}_{\mathcal{C}_B}(M, N).$$

This proposition is an immediate consequence of the following lemma by setting $C = B$, $P = B$:

Lemma 5.2. *Let \mathcal{C} be a k -linear Grothendieck category, and let B, C be k -algebras. Then the following identity holds:*

$$(5.2) \quad \mathrm{RHom}_{C \otimes_k B^{\mathrm{op}}}(P, \mathrm{RHom}_{\mathcal{C}}(M, N)) = \mathrm{RHom}_{\mathcal{C}_C}(P \overset{L}{\otimes}_B M, N)$$

where $M \in D(\mathcal{C}_B)$, $N \in D(\mathcal{C}_C)$, $P \in D(C \otimes_k B^{\mathrm{op}})$.

Proof. We may assume that N is fibrant for the standard model structure on complexes over \mathcal{C}_C (e.g. [3]) and that P is cofibrant as $C \otimes_k B^\circ$ -complex for the projective model structure on complexes [18]. It is easy to see that N is fibrant as a complex over \mathcal{C} and P is cofibrant as a B° -complex. In that case we must show

$$\underline{\mathrm{Hom}}_{C \otimes_k B^\circ}(P, \underline{\mathrm{Hom}}_{\mathcal{C}}(M, N)) = \underline{\mathrm{Hom}}_{\mathcal{C}_C}(P \otimes_B M, N),$$

where $\underline{\mathrm{Hom}}(-, -)$ denotes the morphism complex. We claim the left and right hand side are the same complex. It is enough to show this when P, M, N are objects concentrated in degree zero (with P projective and N injective). In this case we must show

$$\mathrm{Hom}_{C \otimes_k B^\circ}(P, \mathrm{Hom}_{\mathcal{C}}(M, N)) = \mathrm{Hom}_{\mathcal{C}_C}(P \otimes_B M, N).$$

Since P is projective, we may reduce to the case that P is free, so that $P = C \otimes_k B^\circ$. Then we must show

$$\mathrm{Hom}_{C \otimes_k B^\circ}(C \otimes_k B^\circ, \mathrm{Hom}_{\mathcal{C}}(M, N)) = \mathrm{Hom}_{\mathcal{C}_C}((C \otimes_k B^\circ) \otimes_B M, N),$$

which is clearly true. \square

The following is a useful corollary:

Corollary 5.3. *Assuming that M is right bounded and N is left bounded, we get a convergent spectral sequence*

$$E_2^{p,q} = \mathrm{HH}^p(B, \mathrm{Ext}_{\mathcal{C}}^q(M, N)) \Rightarrow \mathrm{Ext}_{\mathcal{C}_B}^{p+q}(M, N).$$

Below we will need the following consequence:

Corollary 5.4. *Assume that \mathcal{C} has global dimension one. Assume furthermore $M, N \in \mathcal{C}_B$. Then there is an isomorphism*

$$\mathrm{HH}^0(B, \mathrm{Hom}_{\mathcal{C}}(M, N)) = \mathrm{Hom}_{\mathcal{C}_B}(M, N)$$

as well as a long exact sequence

$$\begin{aligned} \mathrm{HH}^1(B, \mathrm{Hom}_{\mathcal{C}}(M, N)) &\rightarrow \mathrm{Ext}_{\mathcal{C}_B}^1(M, N) \rightarrow \mathrm{HH}^0(B, \mathrm{Ext}_{\mathcal{C}}^1(M, N)) \rightarrow \\ \mathrm{HH}^2(B, \mathrm{Hom}_{\mathcal{C}}(M, N)) &\rightarrow \mathrm{Ext}_{\mathcal{C}_B}^2(M, N) \rightarrow \mathrm{HH}^1(B, \mathrm{Ext}_{\mathcal{C}}^1(M, N)) \rightarrow \\ \mathrm{HH}^3(B, \mathrm{Hom}_{\mathcal{C}}(M, N)) &\rightarrow \mathrm{Ext}_{\mathcal{C}_B}^3(M, N) \rightarrow \mathrm{HH}^2(B, \mathrm{Ext}_{\mathcal{C}}^1(M, N)) \rightarrow \dots \end{aligned}$$

In particular, if \mathcal{C}_B also has global dimension one then

$$(5.3) \quad \mathrm{HH}^{1+i}(B, \mathrm{Ext}_{\mathcal{C}}^1(M, N)) \cong \mathrm{HH}^{3+i}(B, \mathrm{Hom}_{\mathcal{C}}(M, N))$$

for $i \geq 0$.

Proof. Writing $\mathrm{HH}^p(\mathrm{Ext}^q)$ for $\mathrm{HH}^p(B, \mathrm{Ext}_{\mathcal{C}}^q(M, N))$, the spectral sequence looks like

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & 0 & & \dots \\ & & & & & & & & \\ \mathrm{HH}^0(\mathrm{Ext}^1) & & \mathrm{HH}^1(\mathrm{Ext}^1) & & \mathrm{HH}^2(\mathrm{Ext}^1) & & \mathrm{HH}^3(\mathrm{Ext}^1) & & \dots \\ & \searrow & & \searrow & & \searrow & & & \\ \mathrm{HH}^0(\mathrm{Ext}^0) & & \mathrm{HH}^1(\mathrm{Ext}^0) & & \mathrm{HH}^2(\mathrm{Ext}^0) & & \mathrm{HH}^3(\mathrm{Ext}^0) & & \dots \end{array}$$

The conclusion easily follows. \square

Finally, here is another corollary we will use.

Corollary 5.5. *Assume that there is a k -algebra morphism $\rho : C \rightarrow B$. Let N be in $D(C \otimes_k B^\circ)$. Then there is a canonical isomorphism*

$$\mathrm{RHom}_{B \otimes_k B^\circ}(B, \mathrm{RHom}_C(B, N)) = \mathrm{RHom}_{C \otimes_k C^\circ}(C, N)$$

where we have considered B as a $C \otimes_k B^\circ$ module via the map $\rho \otimes 1 : C \otimes_k B^\circ \rightarrow B \otimes_k B^\circ$.

Proof. We apply (5.1), where $\mathcal{C} = \mathrm{Mod}(C)$, $M = B$. In this way we get

$$\begin{aligned} \mathrm{RHom}_{B \otimes_k B^\circ}(B, \mathrm{RHom}_C(B, N)) &= \mathrm{RHom}_{B^\circ \otimes_k B}(B^\circ, \mathrm{RHom}_C(B, N)) \\ &= \mathrm{RHom}_{C \otimes_k B^\circ}(B, N) \\ &= \mathrm{RHom}_{C \otimes_k C^\circ}(C, N) \end{aligned}$$

The last equality follows from “change of rings” since $(C \otimes_k B^\circ) \overset{L}{\otimes}_{C \otimes_k C^\circ} C = B$. \square

6. LIFTING FIELD ACTIONS IN THE HEREDITARY CASE

Recall that a k -algebra A is defined to be of finite representation type if there are finitely many isomorphism classes of indecomposable left A -modules. A is tame if it is not of finite representation type and if the isomorphism classes of indecomposable left A -modules in any fixed dimension are almost all contained in a finite number of 1-parameter families.

Let A be a finite dimensional k -algebra which is either tame or of finite representation type and let X/k a smooth projective curve of genus $g \leq 1$. Let L/k be an arbitrary field extension.

The principal application of the results in this section is the fact that the essential images of the functors

$$D^b(\mathrm{mod}(A_L)) \rightarrow D^b(\mathrm{Mod}(A))_L$$

and

$$D^b(\mathrm{coh}(X_L)) \rightarrow D^b(\mathrm{Qcoh}(X))_L$$

are precisely the objects which have cohomology in $\mathrm{mod}(A_L) \subset \mathrm{Mod}(A_L) = \mathrm{Mod}(A)_L$ in the first case and in $\mathrm{coh}(X_L) \subset \mathrm{Qcoh}(X_L) = \mathrm{Qcoh}(X)_L$ in the second case.

To be consistent with the setup in the introduction, we would have preferred to talk about $\mathrm{mod}(A)_L$ instead of $\mathrm{mod}(A_L)$ and similarly about $\mathrm{coh}(X)_L$ instead of $\mathrm{coh}(X_L)$. Unfortunately, this is incorrect. If \mathcal{C} is a Hom-finite abelian category and L/k is an infinite field extension, then \mathcal{C}_L contains only the zero object.

In order to be able to describe our results abstractly, we will first discuss a different notion of base extension for essentially small abelian categories such as $\mathrm{mod}(A)$, $\mathrm{coh}(X)$ which behaves in the way we expect.

6.1. Base extension for essentially small abelian categories. Let \mathcal{C} be an essentially small abelian category. The category $\mathrm{Ind} \mathcal{C}$ is obtained by formally closing \mathcal{C} under direct limits (see e.g. [26, §2.2]). It is well known that $\mathrm{Ind} \mathcal{C}$ is a Grothendieck category and, furthermore, \mathcal{C} can be recovered as $\mathrm{Fp} \mathrm{Ind} \mathcal{C}$, the category of finitely presented objects in $\mathrm{Ind} \mathcal{C}$.

A Grothendieck category is said to be locally coherent if it is locally finitely presented (that is: generated by its finitely presented objects) and the finitely

presented objects form an abelian subcategory. Thus $\text{Ind } \mathcal{C}$ is locally coherent. Conversely, if \mathcal{D} is a locally coherent Grothendieck category then $\mathcal{D} \cong \text{Ind Fp } \mathcal{D}$.

Now assume that \mathcal{C} is in addition k -linear, and let B be a k -algebra. If $C \in \mathcal{C}$, then $B \otimes_k C$ is finitely presented in $(\text{Ind } \mathcal{C})_B$, and hence $(\text{Ind } \mathcal{C})_B$ is locally finitely presented. Set

$$\mathcal{C}_{[B]} = \text{Fp}((\text{Ind } \mathcal{C})_B).$$

In good cases, $\mathcal{C}_{[B]}$ will be abelian (or equivalently: $(\text{Ind } \mathcal{C})_B$ will be locally coherent). Here are some typical examples:

- Example 6.1.1.** (1) If X is a k -scheme of finite type, then $\text{coh}(X)_{[L]} = \text{coh}(X_L)$ for L/k an arbitrary field extension.
 (2) If A a finite dimensional k -algebra, then $\text{mod}(A)_{[L]} = \text{mod}(A_L)$.

We need to extend this notion of base extension to the derived setting. Assuming that $\mathcal{C}_{[B]}$ is abelian, we will define $D^b(\mathcal{C})_{[B]}$ as the full subcategory of $D^b(\text{Ind } \mathcal{C})_B$ whose objects have cohomology in $\mathcal{C}_{[B]}$. Thus we have a 2-Cartesian commutative diagram

$$(6.1) \quad \begin{array}{ccc} D^b(\mathcal{C}_{[B]}) & \xhookrightarrow{\quad} & D^b((\text{Ind } \mathcal{C})_B) \\ [F] \downarrow & & \downarrow F \\ D^b(\mathcal{C})_{[B]} & \xhookrightarrow{\quad} & D^b(\text{Ind } \mathcal{C})_B \end{array}$$

The full faithfulness of the lower horizontal arrow is by definition, whereas the full faithfulness of the upper arrow is an application of [26, Prop. 2.14], which asserts that for an essentially small abelian category \mathcal{D} the natural functor

$$D^b(\mathcal{D}) \rightarrow D^b(\text{Ind } \mathcal{D})$$

is fully faithful (and its essential image is $D^b_{\mathcal{D}}(\text{Ind } \mathcal{D})$). We apply this result with $\mathcal{D} = \mathcal{C}_{[B]}$ since then by construction we have $\text{Ind } \mathcal{D} = \text{Ind Fp}((\text{Ind } \mathcal{C})_B) \cong (\text{Ind } \mathcal{C})_B$.

6.2. General discussion. Let \mathcal{D} be a k -linear hereditary category, i.e. an abelian category of global dimension one, and let L/k be a field extension. Let $Z \in D^b(\mathcal{D})_L$. In $D^b(\mathcal{D})$ we have $Z \cong \bigoplus_i s^i H^{-i}(Z)$, where s denotes the shift functor. Thus $\text{End}_{\mathcal{D}}(Z)$ is given by lower triangular matrices

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \text{End}_{\mathcal{D}}(H^{-i+1}(Z)) & 0 & 0 & \dots \\ \dots & \text{Ext}_{\mathcal{D}}^1(H^{-i+1}(Z), H^{-i}(Z)) & \text{End}_{\mathcal{D}}(H^{-i}(Z)) & 0 & \dots \\ \dots & 0 & \text{Ext}_{\mathcal{D}}^1(H^{-i}(Z), H^{-i-1}(Z)) & \text{End}_{\mathcal{D}}(H^{-i-1}(Z)) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Similarly $\text{Ext}_{\mathcal{D}}^j(Z, Z)$ is given by

$$(6.2) \quad \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \text{Hom}_{\mathcal{D}}(H^{-i+1}(Z), H^{-i+1-j}(Z)) & 0 & 0 & \dots \\ \dots & \text{Ext}_{\mathcal{D}}^1(H^{-i+1}(Z), H^{-i-j}(Z)) & \text{Hom}_{\mathcal{D}}(H^{-i}(Z), H^{-i-j}(Z)) & 0 & \dots \\ \dots & 0 & \text{Ext}_{\mathcal{D}}^1(H^{-i}(Z), H^{-i-1-j}(Z)) & \text{End}_{\mathcal{D}}(H^{-i-1}(Z), H^{-i-1-j}(Z)) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The L -bimodule structure on $\text{Ext}_{\mathcal{D}}^j(Z, Z)$ is given by its action on Z , i.e. by its morphism $L \rightarrow \text{End}_{\mathcal{D}}(Z)$.

Definition 6.2.1. Let \mathcal{C} be an essentially small k -linear abelian category which satisfies the following additional conditions for every field extension L/k :

- (E1) $\mathcal{C}_{[L]}$ is abelian.
- (E2) Every object in $\mathcal{C}_{[L]}$ is a direct sum of indecomposable objects.

We say that \mathcal{C} is of essential dimension $\leq d$ if every indecomposable object V in $\mathcal{C}_{[L]}$ with L algebraically closed can be defined over a finitely generated field extension of k of transcendence degree $\leq d$. More precisely, there exists a finitely generated L_0/k such that $\text{trdeg}_k L_0 \leq d$ and $V_0 \in \mathcal{C}_{[L_0]}$ such that $V \cong L \otimes_{L_0} V_0$, where $- \otimes_{L_0} V$ is the unique finite colimit preserving functor such that $L_0 \otimes_{L_0} V = V$. The minimal such d valid for all V is called the essential dimension $\text{ess } \mathcal{C}$ of \mathcal{C} . If such d does not exist then $\text{ess } \mathcal{C} = \infty$.

Note that (E2) holds if $\mathcal{C}_{[L]}$ is Hom-finite.

Remark 6.2.2. • If $\mathcal{C} = \text{mod}(A)$, where A is a finite dimensional hereditary k -algebra, then it follows from the classification of indecomposable representations for Dynkin and extended Dynkin quivers [14, 34] as well as the existence of arbitrarily large moduli spaces in the other cases (e.g. Proposition 3.5.9) that $\text{ess } \mathcal{C} = 0, 1, \infty$ depending on whether \mathcal{C} is of finite representation type, tame or wild.

- If $\mathcal{C} = \text{coh}(X)$, where X is a projective smooth curve/ k , then it follows from the Grothendieck classification of indecomposable coherent sheaves on \mathbb{P}^1 and the corresponding (much harder) classification by Atiyah for elliptic curves, as well as the existence of arbitrarily large moduli spaces in the other cases (e.g. §4), that $\text{ess } \mathcal{C}$ is 1 if X is \mathbb{P}^1 or an elliptic curve and ∞ otherwise.
- It is not clear to us if there can be examples with $\text{ess } \mathcal{C}$ strictly bigger than 1 but finite. In the standard algebraic and geometric cases this is probably excluded by the tame-wild dichotomy.

Theorem 6.2.3. *Let \mathcal{C} be an essentially small k -linear abelian category satisfying (E1)(E2) above, and assume in addition that $\text{Ind } \mathcal{C}$ is hereditary. Consider the usual forgetful functor $[F] : D^b(\mathcal{C}_{[L]}) \rightarrow D^b(\mathcal{C})_{[L]}$ for an arbitrary field extension L/k .*

- *If \mathcal{C} has essential dimension ≤ 2 , $[F]$ is essentially surjective.*
- *If \mathcal{C} has essential dimension ≤ 1 , $[F]$ is in addition full.*
- *If \mathcal{C} has essential dimension 0, $[F]$ is an equivalence of categories.*

Proof. The three parts of the theorem are similar, so we will only prove the first assertion. Assume thus $\text{ess } \mathcal{C} \leq 2$.

Set $\mathcal{D} = \text{Ind } \mathcal{C}$ and let $Z \in D^b(\mathcal{C})_{[L]} \subset D^b(\mathcal{D})_L$. By (6.1) it is sufficient to prove that Z is in the essential image of $D^b(\mathcal{D}_L)$.

The lower triangular structure of the matrix (6.2) equips $\text{Ext}_{\mathcal{D}}^j(Z, Z)$ in a natural way with a two-step filtration stable under the left and right $\text{End}_{\mathcal{D}}(Z)$ -action. Hence $\text{Ext}_{\mathcal{D}}^j(Z, Z)$ is a two-step filtered L -bimodule and the associated quotients are sums of

$$\text{Ext}_{\mathcal{D}}^l(U, V)$$

where U, V are among the $H^i(Z)$. In particular U, V are objects in $\mathcal{C}_{[L]}$.

To prove essential surjectivity, we have to show the vanishing of

$$\text{HH}^n(L, \text{Ext}_{\mathcal{D}}^{-n+2}(Z, Z))$$

for $n \geq 3$ (see Proposition 10.1.1 below). Using the above filtration, it is sufficient to show the vanishing of

$$(6.3) \quad \mathrm{HH}^n(L, \mathrm{Ext}_{\mathcal{D}}^l(U, V))$$

for all $l, n \geq 3$ and for $U, V \in \mathcal{C}_{[L]}$. Let \bar{L} be the algebraic closure of L . We have

$$\begin{aligned} \mathrm{HH}^n(\bar{L}, \mathrm{Ext}_{\mathcal{D}}^l(\bar{L} \otimes_L U, \bar{L} \otimes_L V)) &= \mathrm{HH}^n(\bar{L}, \mathrm{Hom}_{\bar{L}}(\bar{L}, \mathrm{Ext}_{\mathcal{D}}^l(U, \bar{L} \otimes_L V))) \\ &= \mathrm{HH}^n(L, \mathrm{Ext}_{\mathcal{D}}^l(U, \bar{L} \otimes_L V)) \quad (\text{Corollary 5.5}). \end{aligned}$$

Since V is a direct summand of $\bar{L} \otimes_L V$, it suffices to prove that (6.3) vanishes in the case that L is algebraically closed. It is clear that we may assume in addition that U, V are indecomposable. Since $\mathrm{ess} \mathcal{C} \leq 2$ we may write $U = L \otimes_{L_0} U_0$ where L_0/k is a finitely generated field extension of transcendence degree at most two and U_0 is in \mathcal{D}_{L_0} . We then find $\mathrm{Ext}_{\mathcal{D}}^l(U, V) = \mathrm{Hom}_{L_0}(L, \mathrm{Ext}_{\mathcal{D}}^l(U_0, V))$ and so

$$\begin{aligned} \mathrm{HH}^n(L, \mathrm{Ext}_{\mathcal{D}}^l(U, V)) &= \mathrm{HH}^n(L, \mathrm{Hom}_{L_0}(L, \mathrm{Ext}_{\mathcal{D}}^l(U_0, V))) \\ &= \mathrm{HH}^n(L_0, \mathrm{Ext}_{\mathcal{D}}^l(U_0, V)) \quad (\text{Corollary 5.5}) \\ &= 0 \quad (\text{since } n \geq 3) \quad \square \end{aligned}$$

7. COUNTEREXAMPLES TO LIFTING IN THE HEREDITARY CASE

In this section we prove a non-lifting theorem in the hereditary case. In contrast to the previous section we use standard base extension for abelian categories as defined in the introduction.

Proposition 7.1. *Let either \mathcal{C} be $\mathrm{Mod}(kQ)$ with Q be a connected finite “wild” quiver (i.e. Q is not Dynkin or extended Dynkin) or else \mathcal{C} be $\mathrm{Qcoh}(X)$ with X a curve of genus ≥ 2 . Then there exists a finitely generated field extension L/k of transcendence degree 3 together with an object $Z \in D^b(\mathcal{C})_L$ which is not a direct summand of an object in the essential image of the forgetful functor $F : D^b(\mathcal{C}_L) \rightarrow D^b(\mathcal{C})_L$. We may in addition assume that the cohomology of Z lies in $\mathrm{mod}(kQ_L)$ or $\mathrm{coh}(X_L)$, depending on the situation.*

The proof will occupy the rest of this section. For simplicity we will assume that Q is not the quiver with one vertex and two loops, as this case needs a more general argument which we will give in Appendix A.

We first give a necessary and sufficient condition for an object in $D^b(\mathcal{C})_L$ to be in the essential image of F assuming that, after forgetting the L -action, it has the form $Z = U \oplus sV \in D^b(\mathcal{C})$ for $U, V \in \mathcal{C}$. We do this by specializing the general formulas from §6.2. We have

$$\Lambda \stackrel{\mathrm{def}}{=} \mathrm{End}_{\mathcal{C}}(Z) = \begin{pmatrix} \mathrm{End}_{\mathcal{C}}(U) & 0 \\ \mathrm{Ext}_{\mathcal{C}}^1(U, V) & \mathrm{End}_{\mathcal{C}}(V) \end{pmatrix}.$$

An L -action on Z is a k -algebra morphism

$$\phi : L \rightarrow \Lambda.$$

We may write

$$\phi = \begin{pmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

where ϕ_{11}, ϕ_{22} represent an action of L on U and V respectively, so that (U, ϕ_{11}) and (V, ϕ_{22}) are in \mathcal{C}_L . We will denote by (Z, ϕ) the corresponding object of $D^b(\mathcal{C})_L$.

The condition that ϕ is compatible with multiplication yields

$$\begin{aligned} \begin{pmatrix} \phi_{11}(l_1 l_2) & 0 \\ \phi_{21}(l_1 l_2) & \phi_{22}(l_1 l_2) \end{pmatrix} &= \begin{pmatrix} \phi_{11}(l_1) & 0 \\ \phi_{21}(l_1) & \phi_{22}(l_1) \end{pmatrix} \begin{pmatrix} \phi_{11}(l_2) & 0 \\ \phi_{21}(l_2) & \phi_{22}(l_2) \end{pmatrix} \\ &= \begin{pmatrix} \phi_{11}(l_1)\phi_{11}(l_2) & 0 \\ \phi_{22}(l_1)\phi_{21}(l_2) + \phi_{21}(l_1)\phi_{11}(l_2) & \phi_{22}(l_1)\phi_{22}(l_2) \end{pmatrix} \end{aligned}$$

In other words,

$$\phi_{21} : L \rightarrow \text{Ext}_{\mathcal{C}}^1(U, V)$$

must be a k -derivation for the L -bimodule structure on $\text{Ext}_{\mathcal{C}}^1(U, V)$ obtained from the L -structures on U and V .

Lemma 7.2. *(Z, ϕ) as above is in the essential image of F if and only if ϕ_{21} is trivial in*

$$\text{HH}^1(L, \text{Ext}_{\mathcal{C}}^1(U, V)).$$

Proof. We will write ϕ_{triv} for the trivial action on Z coming from the given L -action on U, V (so that $\phi_{21} = 0$).

Assume that (Z, ϕ) is in the essential image of F . In other words, there exists $Y \in D^b(\mathcal{C}_L)$ such that

$$(7.1) \quad (Z, \phi) \cong FY.$$

Since \mathcal{C} is either $\text{Mod}(kQ)$ or $\text{Qch}(X)$, \mathcal{C}_L is of the same type (e.g. $\text{Mod}(kQ)_L = \text{Mod}(LQ)$), hence it is hereditary as well and we have in $D^b(\mathcal{C}_L)$

$$Y \cong \bigoplus_n s^n H^{-n}(Y).$$

Furthermore, in \mathcal{C}_L we have

$$H^{-n}(Y) = H^{-n}(Z) = \begin{cases} U & \text{if } n = 0 \\ V & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus Y , considered as an element of $D^b(\mathcal{C})_L$, is precisely (Z, ϕ_{triv}) . In other words $(Z, \phi) \cong FY$ for some $Y \in D^b(\mathcal{C}_L)$ iff there is an isomorphism in $D^b(\mathcal{C})_L$

$$(7.2) \quad \pi : (Z, \phi) \cong (Z, \phi_{\text{triv}}).$$

We may view π as a unit in Λ , and the condition that π is compatible with the L -action may be expressed as

$$\pi\phi(l) = \phi_{\text{triv}}(l)\pi$$

for all $l \in L$, i.e.

$$(7.3) \quad \phi(l) = \pi^{-1}\phi_{\text{triv}}(l)\pi.$$

We now write all conditions explicitly: we have

$$\begin{aligned} \phi &= \begin{pmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{pmatrix} \\ \phi_{\text{triv}} &= \begin{pmatrix} \phi_{11} & 0 \\ 0 & \phi_{22} \end{pmatrix} \\ \pi &= \begin{pmatrix} \pi_{11} & 0 \\ \pi_{21} & \pi_{22} \end{pmatrix} \end{aligned}$$

$$\pi^{-1} = \begin{pmatrix} \pi_{11}^{-1} & 0 \\ -\pi_{22}^{-1}\pi_{21}\pi_{11}^{-1} & \pi_{22}^{-1} \end{pmatrix}$$

and condition (7.3) translates into

$$\begin{pmatrix} \phi_{11} & 0 \\ \phi_{21} & \phi_{22} \end{pmatrix} = \begin{pmatrix} \pi_{11}^{-1}\phi_{11}\pi_{11} & 0 \\ -\pi_{22}^{-1}\pi_{21}\pi_{11}^{-1}\phi_{11}\pi_{11} + \pi_{22}^{-1}\phi_{22}\pi_{21} & \pi_{22}^{-1}\phi_{22}\pi_{22} \end{pmatrix}$$

which yields

$$\begin{aligned} \pi_{11}\phi_{11} &= \phi_{11}\pi_{11} \\ \pi_{22}\phi_{22} &= \phi_{22}\pi_{22} \\ \phi_{21} &= -\pi_{22}^{-1}\pi_{21}\pi_{11}^{-1}\phi_{11}\pi_{11} + \pi_{22}^{-1}\phi_{22}\pi_{21}. \end{aligned}$$

Taking into account the commutation relation given by the first two relations, the last one can be written as

$$\phi_{21} = \phi_{22}(\pi_{22}^{-1}\pi_{21}) - (\pi_{22}^{-1}\pi_{21})\phi_{11}.$$

In other words, the existence of π implies that ϕ_{21} is an inner derivation, which means precisely that ϕ_{21} is a coboundary in the Hochschild complex. It is easy to see that this implication is reversible. \square

Lemma 7.3. *If $(Z, \phi) \in D^b(\mathcal{C})_L$ as above is not in the essential image of F , then it is also not a direct summand of an object in the essential image of F .*

Proof. Assume that (Z, ϕ) is not in the essential image of F but there exist $W \in D^b(\mathcal{C})_L$, $Y \in D^b(\mathcal{C}_L)$ such that

$$(7.4) \quad (Z, \phi) \oplus W \cong FY.$$

Note that the truncation functors $\tau_{\leq i}$, $\tau_{\geq i}$ commute with F . Applying $\tau_{\leq 0}\tau_{\geq -1}$ to (7.4) we obtain the existence of objects $(Z', \phi') \in D^b(\mathcal{C})_L$ ($Z' = U' \oplus sV'$, $U', V' \in \mathcal{C}$), $Y' \in D^b(\mathcal{C}_L)$ such that $(Z \oplus Z', \phi + \phi') \cong FY'$. This means that $\phi_{21} + \phi'_{21}$ is zero in $\mathrm{HH}^1(L, \mathrm{Ext}_{\mathcal{C}}^1(U \oplus U', V \oplus V'))$. However, it is clear that ϕ_{21} and ϕ'_{21} land in different summands of $\mathrm{HH}^1(L, \mathrm{Ext}_{\mathcal{C}}^1(U \oplus U', V \oplus V'))$. So this implies $\phi_{21} = 0$, which is in contradiction with the fact that (Z, ϕ) is not in the essential image of F . \square

Proof of Proposition 7.1. Now we recall that by (5.3) we have

$$\mathrm{HH}^1(L, \mathrm{Ext}_{\mathcal{C}}^1(U, V)) \cong \mathrm{HH}^3(L, \mathrm{Hom}_{\mathcal{C}}(U, V)).$$

Let $\mathcal{C}_{0,L} = \mathrm{mod}(kQ_L)$ or $\mathrm{coh}(X_L)$, depending on whether \mathcal{C} is equal to $\mathrm{Mod}(kQ)$ or $\mathrm{Qcoh}(X)$. To construct Z as in the statement of the proposition, it suffices by Lemmas 7.2 and 7.3 to find L and $U, V \in \mathcal{C}_{0,L}$ such that $\mathrm{HH}^3(L, \mathrm{Hom}_{\mathcal{C}}(U, V)) \neq 0$. We will in fact produce a finitely generated field extension L/k of transcendence degree 3 and $U \in \mathcal{C}_{0,L}$ such that $\mathrm{End}_{\mathcal{C}}(U) = L$, and let $V = U$. This will do what we want by the Hochschild-Kostant-Rosenberg theorem [17].

Let us first consider the case $\mathcal{C} = \mathrm{Mod}(kQ)$, Q not the two-loop quiver. Choose α indivisible, and in the fundamental region as in Proposition 3.5.9, such that $\dim M_{\alpha,\lambda,Q} \geq 3$. Let x be the generic point of a three dimensional irreducible subvariety of $M_{\alpha,\lambda,Q}$, and put $L = k(x)$. Let $U = V_x$ as in Proposition 3.5.2. Then according to (3.3) we have indeed $\mathrm{End}_{kQ}(U) = L$.

Next consider $\mathcal{C} = \mathrm{Qcoh}(X)$. We choose r, d , $\gcd(r, d) = 1$ such that $\dim M_{r,d} \geq 3$. Then we proceed as in the case $\mathcal{C} = \mathrm{Mod}(kQ)$, but now using Proposition 4.1. \square

8. COUNTEREXAMPLES TO LIFTING IN THE GEOMETRIC CASE

In this section we will prove the following result:

Theorem 8.1. *Let Y/k be a smooth connected projective variety which is not a point, a projective line or an elliptic curve. Then there exists a finitely generated field extension L/k of transcendence degree 3 together with an object $Z \in D^b(\text{Qcoh}(Y))_L$ which is not a direct summand of an object in the essential image of the forgetful functor $F : D^b(\text{Qcoh}(Y)_L) \rightarrow D^b(\text{Qcoh}(Y))_L$. We may in addition assume that the cohomology of Z lies in $\text{coh}(Y_L)$.*

Proof. If Y is a curve then this follows from Proposition 7.1, so we may assume $\dim Y \geq 2$. We start by considering the case $Y = \mathbb{P}^d$, $d \geq 2$. Put $T = \mathcal{O}_Y \oplus \mathcal{O}_Y(1)$ and $A = \text{End}_Y(T)^\circ$. Then T is a partial tilting object and we have functors

$$D^b(\text{Mod}(A)) \xrightarrow[i]{T \otimes_A^-} D^b(\text{Qcoh}(Y)) \xrightarrow[j]{\text{RHom}_Y(T, -)} D^b(\text{Mod}(A))$$

such that ji is the identity. We can define analogous functors, also denoted by i and j , on $D^b(\text{Mod}(A_L))$, $D^b(\text{Mod}(A)_L)$, etc., and i, j commute with the functor F . Now A is the path algebra of a Kronecker quiver with $d+1 \geq 3$ arrows, and so according to Proposition 7.1 there exists a finitely generated field extension L/k of transcendence degree three and an object $Z_0 \in D^b(\text{Mod}(A))_L$ with cohomology in $\text{mod}(A_L)$ which is not a direct summand of the image of an object in $D^b(\text{Mod}(A)_L)$. Put $Z_1 = i(Z_0) \in D^b(\text{Qcoh}(Y))_L$. Clearly Z_1 has cohomology in $\text{coh}(Y_L)$. If there exists $Z' \in D^b(\text{Qcoh}(Y))_L$, $Y \in D^b(\text{Qcoh}(Y)_L)$ such that $Z_1 \oplus Z' = FY$, then applying j we get $Z_0 \oplus j(Z') = F(j(Y))$ in $D^b(\text{Mod}(A)_L)$, which we had excluded. This finishes the proof in the case $Y = \mathbb{P}^d$.

Now let Y be general, choose a finite (necessarily flat) map $\pi : Y \rightarrow \mathbb{P}^d$, and let $Z_1 \in D^b(\text{Qcoh}(\mathbb{P}^d))_L$ be as above. Set $Z = \pi^*Z_1$. If there exists $Z' \in D^b(\text{Qcoh}(Y))_L$, $Y \in D^b(\text{Qcoh}(Y)_L)$ such that $Z \oplus Z' = FY$ then applying $R\pi_*$ we get $R\pi_*\pi^*Z_1 \oplus R\pi_*Z' = F(R\pi_*Y)$. Now $R\pi_*\pi^*Z_1 = \pi_*\mathcal{O}_Y \otimes_{\mathcal{O}_{\mathbb{P}^d}} Z_1$ and $\mathcal{O}_{\mathbb{P}^d}$ is a direct summand of $\pi_*\mathcal{O}_Y$ in $\text{coh}(\mathbb{P}^d)$ (by the trace map). Hence Z_1 is a direct summand of $R\pi_*\pi^*Z_1$ in $D^b(\text{Qcoh}(\mathbb{P}^d))_L$, and thus Z_1 is also a direct summand of $F(R\pi_*Y)$ in $D^b(\text{Qcoh}(\mathbb{P}^d))_L$. This is a contradiction with the choice of Z_1 . \square

Example 8.2. Assume that $Y = \mathbb{P}_k^3$, $L = k(x, y, z)$ and let $p : \mathbb{P}_L^3 \rightarrow \mathbb{P}_k^3$ be obtained by base extension of the structure map $\text{Spec } L \rightarrow \text{Spec } k$. If we construct Z_0 using the object V_η in Example 3.5.6 (see the proof of Proposition 7.1) then we find that, after forgetting the L -structure, $Z \cong p_*R \oplus p_*sR$, where R is given by

$$\text{cone}(\mathcal{O}_{\mathbb{P}_L^3}(-1)^3 \xrightarrow{(Tx-X, Ty-Y, Tz-Z)} \mathcal{O}_{\mathbb{P}_L^3})$$

and T, X, Y, Z are homogeneous coordinates on \mathbb{P}^3 .

9. NON-FOURIER-MUKAI FUNCTORS

Below X, Y are smooth connected projective schemes over k , although we could get by with substantially less. Let $i_\eta : \eta \rightarrow X$ be the generic point and let $L = k(\eta)$ be the function field of X . Assume that $D^b(\text{Qcoh}(Y))_L$ contains an object Z which is not in the essential image of $D^b(\text{Qcoh}(Y)_L)$. Define the exact functor

$\tilde{\Psi} : D(\text{Qcoh}(X)) \rightarrow D(\text{Qcoh}(Y))$ as the composition

$$\tilde{\Psi} : D(\text{Qcoh}(X)) \xrightarrow{i_\eta^*} D(L) \xrightarrow{\psi} D(\text{Qcoh}(Y))$$

where $\psi : D(L) \rightarrow D(\text{Qcoh}(Y))$ is the unique additive functor commuting with shifts and coproducts which sends L to Z , and is determined on morphisms by the structure of Z as an L -object. This functor is exact, because L is a field. By construction, $\tilde{\Psi}$ commutes with coproducts. Let $\Psi : \text{Perf}(X) \rightarrow D^b(\text{Qcoh}(Y))$ be the restriction of $\tilde{\Psi}$ to $\text{Perf}(X) = D^b(\text{coh}(X))$.

Theorem 9.1. *The functor*

$$\Psi : \text{Perf}(X) \rightarrow D^b(\text{Qcoh}(Y))$$

as defined above is not the restriction of a Fourier-Mukai functor $D(\text{Qcoh}(X)) \rightarrow D(\text{Qcoh}(Y))$ associated to an object in $D(\text{Qcoh}(X \times Y))$.

Taking Y, Z, L as in Theorem 8.1, and letting X be a smooth projective model for L , i.e. such that $K(X) = L$, gives a counterexample to Proposition A in the introduction if we drop the condition that Ψ is fully faithful. By taking Z as in Example 8.2 we get a counterexample where $X = Y = \mathbb{P}^3$.

We will give the proof of Theorem 9.1 below, after some preparatory lemmas.

Lemma 9.2. *Assume that*

$$\Phi : D(\text{Qcoh}(X)) \rightarrow D(\text{Qcoh}(Y))$$

is an exact functor, commuting with coproducts, whose restriction to $\text{Perf}(X)$ is naturally equivalent to Ψ . Then Φ is naturally equivalent to $\tilde{\Psi}$.

Proof. We first claim that Φ factors uniquely as

$$(9.1) \quad D(\text{Qcoh}(X)) \xrightarrow{i_\eta^*} D(L) \xrightarrow{\phi} D(\text{Qcoh}(Y)),$$

where ϕ is an exact functor commuting with coproducts. To see this note that the first arrow i_η^* is a Verdier localization at the full subcategory \mathcal{C} of $D(\text{Qcoh}(X))$ spanned by objects \mathcal{M} such that $i_\eta^* \mathcal{M} = 0$. We claim that \mathcal{C} is generated by objects which are compact in $D(\text{Qcoh}(X))$ (i.e. perfect complexes).

By following the inductive procedure of the proof of [28, Prop. 2.5] (also [4, Thm 3.1.1]) one reduces this claim to the case that $X = \text{Spec } R$. In that case $D(\text{Qcoh}(X)) = D(R)$ and L is the quotient field of R . The complexes $M(s) = R \xrightarrow{s \times} R$, $s \in R - \{0\}$ are generators for the kernel of $D(R) \xrightarrow{L \otimes_R -} D(L)$. Indeed if N is in this kernel and is right orthogonal to all shifts of all $M(s)$ for $s \in R - \{0\}$ then for all such s and all i the map $H^i(N) \xrightarrow{s \times} H^i(N)$ is an isomorphism. On the other hand since $H^i(N)$ is annihilated by L , a non-zero element of $H^i(N)$ is annihilated by some $s \in R - \{0\}$ which is impossible. Thus $H^i(N) = 0$ for all i and hence $N = 0$.

Since the restriction of Φ to the perfect complexes in \mathcal{C} is equal to Ψ and since Ψ annihilates such complexes and furthermore since Φ preserves coproducts, Φ vanishes on \mathcal{C} . Thus the asserted factorization (9.1) follows.

Thus it suffices to prove that ψ and ϕ are naturally equivalent, given that we know that $\psi \circ i_\eta^*$ and $\phi \circ i_\eta^*$ are naturally equivalent when restricted to $\text{Perf}(X)$. For this we must show that $\psi(L)$ and $\phi(L)$ are isomorphic as L -objects in $D(\text{Qcoh}(Y))$.

Now we have $L = i_\eta^*(\mathcal{O}_X)$, and so

$$\psi(L) = (\psi \circ i_\eta^*)(\mathcal{O}_X) \cong (\phi \circ i_\eta^*)(\mathcal{O}_X) = \phi(L).$$

So we certainly have an isomorphism $\sigma : \psi(L) \cong \phi(L)$ in $D(\text{Qcoh}(Y))$. To prove that this isomorphism is compatible with the L -structure, we observe that any map $f : L \rightarrow L$ is of the form $i_\eta^*(g) \circ i_\eta^*(h)^{-1}$ where g, h are morphisms $\mathcal{O}_X(-nE) \rightarrow \mathcal{O}_X$ with E an ample divisor and h non-zero. Thus we get a diagram

$$(9.2) \quad \begin{array}{ccccc} & & \sigma & & \\ & \nearrow & & \searrow & \\ \psi(L) & \xlongequal{\quad} & (\psi \circ i_\eta^*)(\mathcal{O}_X) & \xrightarrow{\cong} & (\phi \circ i_\eta^*)(\mathcal{O}_X) & \xlongequal{\quad} & \phi(L) \\ & \downarrow & \downarrow (\psi \circ i_\eta^*)(h)^{-1} & & \downarrow (\phi \circ i_\eta^*)(h)^{-1} & & \downarrow \phi(f) \\ & \psi(f) & (\psi \circ i_\eta^*)(\mathcal{O}_X(-nE)) & \xrightarrow{\cong} & (\phi \circ i_\eta^*)(\mathcal{O}_X(-nE)) & & \\ & & \downarrow (\psi \circ i_\eta^*)(g) & & \downarrow (\phi \circ i_\eta^*)(g) & & \\ \psi(L) & \xlongequal{\quad} & (\psi \circ i_\eta^*)(\mathcal{O}_X) & \xrightarrow{\cong} & (\phi \circ i_\eta^*)(\mathcal{O}_X) & \xlongequal{\quad} & \phi(L) \\ & & \sigma & & & & \end{array}$$

where:

- (1) the leftmost rectangle is commutative, since it is obtained by applying ψ to $f = i_\eta^*(g) \circ i_\eta^*(h)^{-1}$;
- (2) the rightmost rectangle is commutative for the same reason;
- (3) the lower middle rectangle is commutative, since it is obtained from the natural isomorphism $\psi \circ i_\eta^* \cong \phi \circ i_\eta^*$;
- (4) the upper middle rectangle is commutative, since it is obtained from inverting the vertical arrows in the commutative diagram

$$\begin{array}{ccc} (\psi \circ i_\eta^*)(\mathcal{O}_X) & \xrightarrow{\cong} & (\phi \circ i_\eta^*)(\mathcal{O}_X) \\ \uparrow (\psi \circ i_\eta^*)(h) & & \uparrow (\phi \circ i_\eta^*)(h) \\ (\psi \circ i_\eta^*)(\mathcal{O}_X(-nE)) & \xrightarrow{\cong} & (\phi \circ i_\eta^*)(\mathcal{O}_X(-nE)). \end{array}$$

It follows that the outer rectangle in (9.2) is commutative, and hence σ is indeed compatible with the L -structure. \square

Lemma 9.3. *Assume that*

$$\Phi : D(\text{Qcoh}(X)) \rightarrow D(\text{Qcoh}(Y))$$

is a Fourier-Mukai functor. Then the L object $(\Phi \circ i_{\eta})(L)$ in $D(\text{Qcoh}(Y))$ lies in the essential image of $F : D(\text{Qcoh}(Y))_L \rightarrow D(\text{Qcoh}(Y))_L$.*

Proof. Assume that Φ is isomorphic to the Fourier-Mukai functor Φ_V with kernel $V \in D(\text{Qcoh}(X \times Y))$, i.e. $\Phi_V = Rp_{2*}(V \overset{L}{\otimes} Lp_1^*(\cdot))$. Consider the object $V_\eta \in D(\text{Qcoh}(\text{Spec } L \times Y))$ given by $V_\eta = (i_\eta \times \text{id})^*V$. Below we show that there is a natural isomorphism

$$(9.3) \quad \Phi_V \circ i_{\eta*} \cong \Phi_{V_\eta}$$

as functors $D(L) \rightarrow D(\text{Qcoh}(Y))$.

So it suffices to show that the L object $\Phi_{V_\eta}(L) = Rp_{2*}V_\eta$ in $D(\mathrm{Qcoh}(Y))$ lies in the essential image of F . Now there is a canonical identification $c : \mathrm{Qcoh}(\mathrm{Spec} L \times Y) \rightarrow \mathrm{Qcoh}(Y)_L$ which fits in a commutative diagram

$$\begin{array}{ccc} D(\mathrm{Qcoh}(\mathrm{Spec} L \times Y)) & \xrightarrow{Rp_{2*}} & D(\mathrm{Qcoh}(Y))_L \\ c \downarrow & & \parallel \\ D(\mathrm{Qcoh}(Y)_L) & \xrightarrow{F} & D(\mathrm{Qcoh}(Y))_L. \end{array}$$

Thus we find $Rp_{2*}V_\eta = F(cV_\eta)$ which proves what we want.

Now we verify (9.3). Consider the morphisms

$$\begin{array}{ccccc} & & D(\mathrm{Qcoh}(X \times Y)) & & \\ & \nearrow Lp_1^* & \uparrow R(i_\eta \times \mathrm{id})_* & \searrow Rp_{2*} & \\ & & D(\mathrm{Qcoh}(\mathrm{Spec} L \times Y)) & \xrightarrow{Rp_{2*}} & D(\mathrm{Qcoh}(Y)) \\ & \nwarrow Lp_1^* & & & \\ D(\mathrm{Qcoh}(X)) & \xleftarrow{i_{\eta*}} & D(L) & & \end{array}$$

We have

$$\begin{aligned} (\Phi_V \circ i_{\eta*})(-) &= Rp_{2*}(Lp_1^*(i_{\eta*}(-)) \otimes^L V) \\ &= Rp_{2*}((i_\eta \times \mathrm{id})_*(Lp_1^*(-)) \otimes^L V) \\ &= Rp_{2*}(i_\eta \times \mathrm{id})_*(Lp_1^*(-) \otimes^L (i_\eta \times \mathrm{id})^*V) \\ &= Rp_{2*}(Lp_1^*(-) \otimes^L (i_\eta \times \mathrm{id})^*V) \\ &= \Phi_{V_\eta}(-) \end{aligned}$$

The second equality is flat base change for $p_1 : X \times Y \rightarrow X$. The third equality is the projection formula [24, Prop. 3.9.4] for $i_\eta \times \mathrm{id}$. \square

Proof of Theorem 9.1. Assume that Ψ is the restriction of a Fourier-Mukai functor $\Phi : D(\mathrm{Qcoh}(X)) \rightarrow D(\mathrm{Qcoh}(Y))$. According to Lemma 9.2 we have $\Phi \cong \tilde{\Psi}$. According to Lemma 9.3 $(\Phi \circ i_{\eta*})(L) \cong (\tilde{\Psi} \circ i_{\eta*})(L)$ is in the essential image of $D(\mathrm{Qcoh}(Y))_L$. But since $(\tilde{\Psi} \circ i_{\eta*})(L) = (\psi \circ i_\eta^* \circ i_{\eta*})(L) = \psi(L) = Z$, this is a contradiction. \square

10. LIFTING USING A_∞ -ACTIONS

From now on we only assume that k is a field. We will prove Proposition B stated in the introduction. The results from this section were also used in the proof of Theorem 6.2.3. For the benefit of the reader we will provide some preliminary material concerning A_∞ -actions.

10.1. Introduction. A graded category \mathcal{A} is a category enhanced in the category of graded k -vector spaces. To stress the grading we will sometimes write $\underline{\text{Hom}}_{\mathcal{A}}(-, -)$ to denote the Hom-spaces. We denote the part of degree zero of $\underline{\text{Hom}}_{\mathcal{A}}(-, -)$ by $\text{Hom}_{\mathcal{A}}(-, -)$.

Let \mathcal{C} be a k -linear Grothendieck category. The category of complexes over \mathcal{C} (denoted by $C(\mathcal{C})$) is a DG-category, and, in particular, a graded category. To simplify the notation we write $\underline{\text{Hom}}_{\mathcal{C}}$ for $\underline{\text{Hom}}_{C(\mathcal{C})}$ to denote the morphism complex, and similarly for $\text{Hom}_{\mathcal{C}}$ for $\text{Hom}_{C(\mathcal{C})}$. Let B be a DG-algebra over k (at first reading one may assume that B is just an algebra, concentrated in degree zero). We define the DG-category $C(B, \mathcal{C})$ as the category of pairs (M, ρ_M) , where $M \in C(B, \mathcal{C})$ and $\rho_M : B \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, M)$ is a DG-algebra morphism giving the B -action on M . We put

$$D(B, \mathcal{C}) = Z^0(C(B, \mathcal{C}))[\text{Qis}^{-1}].$$

The construction of $D(B, \mathcal{C})$ represents no set-theoretic difficulties since it may be obtained from a model structure on $C(B, \mathcal{C})$ [25, Prop. 5.1]⁴. $D(B, \mathcal{C})$ can be identified with $D(\mathcal{C}_B)$ when B is concentrated in degree zero.

If \mathcal{A} is an arbitrary graded category and B is a graded k -algebra, we may define the category \mathcal{A}_B whose objects are the objects in \mathcal{A} equipped with a B -action.

Let us go back to the case of a DG-algebra B over k . There is an obvious functor

$$F : D(B, \mathcal{C}) \rightarrow D(\mathcal{C})_{H^*(B)},$$

where $H^*(B)$ is the graded k -algebra $\bigoplus_{i \in \mathbb{Z}} H^i(B)$, with the multiplication induced by the multiplication on B . The functor F is obtained by noticing that $\rho_M : B \rightarrow \underline{\text{Hom}}_{D(\mathcal{C})}(M, M)$ factors through $H^*(B)$ since coboundaries are homotopic to zero.

Below we give proofs of the following results:

Proposition 10.1.1. *Let $M \in D(\mathcal{C})_{H^*(B)}$ be such that for $n \geq 3$*

$$\text{HH}^n(H^*(B), \text{Ext}_{\mathcal{C}}^*(M, M))_{-n+2} = 0.$$

Then there exists an object \tilde{M} in $D(B, \mathcal{C})$ together with an isomorphism $F(\tilde{M}) \cong M$ in $D(\mathcal{C})_{H^(B)}$.*

Proposition 10.1.2. *Let $M, N \in D(B, \mathcal{C})$ be such that for $n \geq 2$*

$$\text{HH}^n(H^*(B), \text{Ext}_{\mathcal{C}}^*(M, N))_{-n+1} = 0.$$

Then the map

$$\text{Hom}_{D(B, \mathcal{C})}(M, N) \rightarrow \text{Hom}_{D(\mathcal{C})_{H^*(B)}}(FM, FN)$$

is surjective.

Proposition 10.1.3. *Let $M, N \in D(B, \mathcal{C})$ be such that for $n \geq 1$*

$$\text{HH}^n(H^*(B), \text{Ext}_{\mathcal{C}}^*(M, N))_{-n} = 0.$$

Then the map

$$\text{Hom}_{D(B, \mathcal{C})}(M, N) \rightarrow \text{Hom}_{D(\mathcal{C})_{H^*(B)}}(FM, FN)$$

is injective.

These results imply Proposition B in the introduction.

10.2. Reminder on A_{∞} -algebras and morphisms.

⁴The proof of this result is based on [3].

10.2.1. *A_∞ -algebras.* Let A be a graded vector space. We denote by $\mathbb{B}A = \bigoplus_{n \geq 1} (sA)^{\otimes n}$ the tensor coalgebra (without counit) of sA . Sometimes we write $sa_1 \otimes \cdots \otimes sa_n \in \mathbb{B}A$ as a tuple (sa_1, \dots, sa_n) . With this convention, the comultiplication is given by

$$\Delta(sa_1, \dots, sa_n) = \sum_{i=1}^{n-1} (sa_1, \dots, sa_i) \otimes (sa_{i+1}, \dots, sa_n).$$

By definition, an A_∞ -structure on A is given by a (graded) coderivation $b : \mathbb{B}A \rightarrow \mathbb{B}A$ of degree 1 and square zero. Thus

$$\begin{aligned} \Delta \circ b &= (b \otimes \text{id} + \text{id} \otimes b) \circ \Delta \\ b^2 &= 0 \end{aligned}$$

The coderivation b is determined by its Taylor coefficients $(b_n)_{n \geq 1}$, which are the compositions

$$(sA)^{\otimes n} \xrightarrow{\text{inclusion}} \mathbb{B}A \xrightarrow{b} \mathbb{B}A \xrightarrow{\text{projection}} sA.$$

The fact that b is a coderivation implies

$$b = \sum_{p,q,r \geq 0} \text{id}^{\otimes p} \otimes b_q \otimes \text{id}^{\otimes r}.$$

Corresponding to the b_n we have the more traditional operations

$$m_n : A^{\otimes n} \rightarrow A$$

of degree $2 - n$, which are related to the b_n by the formula

$$b_n = s^{-n+1} m_n.$$

Explicitly, in the cases $n = 1$ and $n = 2$

$$\begin{aligned} (10.1) \quad b_1(sa) &= -sm_1(a) \\ b_2(sa, sb) &= (-1)^{|a|} sm_2(a, b). \end{aligned}$$

A DG-algebra is the same as an A_∞ -algebra with $b_n = 0$ for $n \geq 3$.

10.2.2. *A_∞ -morphisms.* If A, C are A_∞ -algebras, an A_∞ -morphism $\psi : A \rightarrow C$ is by definition a graded coalgebra morphism $\psi : \mathbb{B}A \rightarrow \mathbb{B}C$ commuting with the differentials. Thus

$$\begin{aligned} \Delta \circ \psi &= (\psi \otimes \psi) \circ \Delta \\ b_C \circ \psi &= \psi \circ b_A \end{aligned}$$

Again ψ is determined by its Taylor coefficients $(\psi_n)_{n \geq 1}$, which are the compositions

$$(sA)^{\otimes n} \xrightarrow{\text{inclusion}} \mathbb{B}A \xrightarrow{\psi} \mathbb{B}C \xrightarrow{\text{projection}} sC.$$

This time we have

$$\psi = \sum_{r, n_1, \dots, n_r} \psi_{n_1} \otimes \psi_{n_2} \otimes \cdots \otimes \psi_{n_r}.$$

There are no sign issues since all ψ_n have degree zero. For this reason we will identify $\psi_1 : sA \rightarrow sC$ with a map $\psi_1 : A \rightarrow C$ (thus $\psi_1(sa) = s\psi_1(a)$).

10.2.3. *A_∞ -modules.* We will define A_∞ -modules over a k -linear Grothendieck category \mathcal{C} , which we fix throughout. If A is an A_∞ -algebra, an A_∞ - A -module in $C(\mathcal{C})$ is an object $M \in C(\mathcal{C})$ together with an A_∞ -morphism $A \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, M)$. Alternatively, define

$$\mathbb{B}M = (\mathbb{B}A)^+ \otimes_k M,$$

where $(-)^+$ means adjoining a counit: $(\mathbb{B}A)^+ = \bigoplus_{n \geq 0} (sA)^{\otimes n}$ and $\Delta(sa_1, \dots, sa_n) = \sum_{i=0}^n (sa_1, \dots, sa_i) \otimes (sa_{i+1}, \dots, sa_n)$ where empty parentheses are to be interpreted as 1. Then $\mathbb{B}M$ is a left $\mathbb{B}A$ -comodule via

$$\Delta_M(sa_1, \dots, sa_n, m) = \sum_{i=1}^n (sa_1, \dots, sa_i) \otimes (sa_{i+1}, \dots, sa_n, m)$$

and an A_∞ -structure on M is given by a $\mathbb{B}A$ coderivation b_M on $\mathbb{B}M$ of degree one and square zero. Thus b_M satisfies

$$\begin{aligned} \Delta_M \circ b_M &= (b_A \otimes \text{id} + \text{id} \otimes b_M) \circ \Delta_M \\ b_M^2 &= 0 \end{aligned}$$

Needless to say, b_M is again determined by its Taylor coefficients, which are morphisms

$$b_{M,n} : (sA)^{\otimes n-1} \otimes_k M \rightarrow M$$

in $C(\mathcal{C})$. We have

$$b_M = \sum_{p,q,r \geq 0} \text{id}_A^{\otimes p} \otimes b_{A,q} \otimes \text{id}_A^{\otimes r} \otimes \text{id}_M + \sum_{m,n \geq 0} \text{id}_A^{\otimes m} \otimes b_{M,n}.$$

We denote the category of A_∞ - A -modules in $C(\mathcal{C})$, with morphisms given by morphisms of complexes, by $C_\infty^{\text{strict}}(A, \mathcal{C})$. It is easy to see that this is a Grothendieck category.

10.2.4. *A_∞ -morphisms between A_∞ -modules.* Let A be an A_∞ -algebra and let $M, N \in C(\mathcal{C})$ be two A_∞ - A -modules in \mathcal{C} . An A_∞ -morphism $\psi : M \rightarrow N$ is a comodule morphism of degree zero $\psi : \mathbb{B}M \rightarrow \mathbb{B}N$ satisfying $b_N \circ \psi = \psi \circ b_M$. Thus ψ satisfies

$$\Delta \circ \psi = (\text{id} \otimes \psi) \circ \Delta.$$

This time, the Taylor coefficients are

$$\psi_n : (sA)^{\otimes n-1} \otimes_k M \rightarrow N$$

and ψ is given by

$$\psi = \sum_{p,q \geq 0} \text{id}^{\otimes p} \otimes \psi_q.$$

We denote the category of A_∞ - A -modules in $C(\mathcal{C})$ equipped with A_∞ -morphisms by $C_\infty(A, \mathcal{C})$.

A homotopy between A_∞ -morphisms $\psi_1, \psi_2 : M \rightarrow N$ is a comodule morphism $h : \mathbb{B}M \rightarrow \mathbb{B}N$ of degree -1 such that $\psi_1 - \psi_2 = b_N \circ h + h \circ b_M$.

10.2.5. *Units.* Let A be an A_∞ -algebra. We say that A has a homological unit if $H^*(A)$ has a unit element 1_A . Let $M \in C_\infty(A, \mathcal{C})$. We say that M is homologically unital if 1_A acts as the identity on $H^*(M)$. All constructions for A_∞ -algebras outlined above have a unital analogue in which we require that on the level of cohomology the units behave as expected. We write $C_\infty^{\text{hu}, \text{strict}}(A, \mathcal{C})$, $C_\infty^{\text{hu}}(A, \mathcal{C})$ for the corresponding categories. Furthermore we put

$$\begin{aligned} D_\infty^{\text{strict}}(A, \mathcal{C}) &= C_\infty^{\text{hu}, \text{strict}}(A, \mathcal{C})[\text{Qis}^{-1}] \\ D_\infty(A, \mathcal{C}) &= C_\infty^{\text{hu}}(A, \mathcal{C})[\text{Qis}^{-1}] \end{aligned}$$

It follows in the usual way that homotopic maps $\psi_1, \psi_2 : M \rightarrow N$ in $C_\infty(A, \mathcal{C})$ yield equal maps in $D_\infty^{\text{strict}}(A, \mathcal{C})$ and $D_\infty(A, \mathcal{C})$.

Lemma 10.2.1. *Let A be a DG-algebra. Then the natural functors*

$$(10.2) \quad D(A, \mathcal{C}) \rightarrow D_\infty^{\text{strict}}(A, \mathcal{C})$$

$$(10.3) \quad D(A, \mathcal{C}) \rightarrow D_\infty(A, \mathcal{C})$$

are equivalences of categories.

Proof. The proof in the strict and non-strict cases is the same, so we consider only (10.3). If \mathcal{C} is the category of k -vector spaces and we restrict ourselves to so-called “strictly unital”⁵ modules, this is [23, Lemme 4.1.3.8]. The proof in loc. cit. goes more or less through in our setting. The first step is the definition of a functor

$$A \otimes_A^\infty - : C_\infty(A, \mathcal{C}) \rightarrow C(A, \mathcal{C}).$$

This definition is given in [23, Lemme 4.1.1.6]. The next step is to prove that $A \otimes_A^\infty -$ yields a quasi-inverse to (10.3) after inverting quasi-isomorphisms. This is part of the proof of [23, Lemme 4.1.1.6]. Ultimately it reduces to (the well-known) Lemma 10.2.2 below. \square

Lemma 10.2.2. *Assume that A is a homologically unital A_∞ -algebra and $M \in C_\infty^{\text{hu}}(A, \mathcal{C})$. Then $\mathbb{B}M$ is acyclic.*

Proof. The fact that $\mathbb{B}M$ is acyclic is proved in [23, Lemme 4.1.1.6], under the hypothesis that A and M are “strictly unital”, by providing an explicit contracting homotopy. If we only assume that A, M are homologically unital then we cannot use this argument.

So we proceed differently. We have to show that $H^*(\mathbb{B}M) = 0$. We define an ascending filtration on $\mathbb{B}M$

$$F_n \mathbb{B}M = \bigoplus_{m \leq n} (sA)^{\otimes m} \otimes M$$

and we consider the resulting spectral sequence. One checks that the first page of this spectral sequence is

$$\mathbb{B}H^*(M)$$

where we consider $H^*(M)$ as an A_∞ -module over $H^*(A)$ with $b_i = 0$ for $i \neq 2$. Since $H^*(A)$ has a true unit and $H^*(M)$ is truly unital, it is well-known that $\mathbb{B}H^*(M)$ is acyclic (for example by using contracting homotopy given in the proof of [23, Lemme 4.1.1.6] alluded to above). \square

⁵An A_∞ -module is strictly unital if $b_{M,n}$, with $n \geq 2$, vanishes as soon as one of the arguments is 1 and $b_{2,M}(\text{id}_M \otimes 1_A) = \text{id}_M$ where η is the unit of A .

10.3. Proof of Proposition 10.1.1.

Lemma 10.3.1. *Let A, C be two A_∞ -algebras over k and let $\phi : A \rightarrow C$ be a k -linear map commuting with the differentials m_1 such that $H^*(\phi) : H^*(A) \rightarrow H^*(C)$ is a graded algebra morphism. Assume that for all $n \geq 3$ we have*

$$\mathrm{HH}^n(H^*(A), H^*(C))_{-n+2} = 0.$$

Then there exists an A_∞ -morphism $\psi : A \rightarrow C$ such that $\psi_1 = \phi$.

Proof. This is close to the obstruction theory in [23, Appendix B.4] for minimal A_∞ -algebras. Rather than reducing to it by invoking the fact that any A_∞ -algebra is A_∞ -homotopy equivalent to a minimal one, we give a simple direct proof for the benefit of the reader.

We will construct ψ step by step. We first put $\psi_1 = \phi$. ϕ is compatible with the multiplications on A and C up to a homotopy, which we take to be $-\psi_2$. Thus

$$(10.4) \quad b_{C,2} \circ (\psi_1 \otimes \psi_1) - \psi_1 \circ b_{A,2} = -b_{C,1} \circ \psi_2 + \psi_2 \circ (b_{A,1} \otimes \mathrm{id} + \mathrm{id} \otimes b_{A,1}).$$

Assume that we have constructed ψ_1, \dots, ψ_n . Let $\psi_{\leq n} : \mathbb{B}A \rightarrow \mathbb{B}C$ be the coalgebra map such that

$$(\psi_{\leq n})_i = \begin{cases} \psi_i & i = 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

Assume furthermore that

$$(10.5) \quad b_C \circ \psi_{\leq n} = \psi_{\leq n} \circ b_A \quad \text{restricted to } (sA)^{\otimes i} \text{ for } 1 \leq i \leq n.$$

It follows from (10.4) that we have already achieved this for $n \leq 2$.

Our aim is to construct ψ_{n+1} such that $b_C \circ \psi_{\leq n+1} = \psi_{\leq n+1} \circ b_A$ when restricted to $(sA)^{\otimes i}$ for $1 \leq i \leq n+1$. Before we start, we warn the reader that the construction of ψ_{n+1} will involve changing ψ_n .

Consider

$$(10.6) \quad D = b_C \circ \psi_{\leq n} - \psi_{\leq n} \circ b_A.$$

Then $D : \mathbb{B}A \rightarrow \mathbb{B}C$ is a $\psi_{\leq n}$ coderivation of degree 1. By construction we have $D_m = 0$ for $m = 1, \dots, n$. Moreover, it is clear that we have

$$(10.7) \quad b_C \circ D + D \circ b_A = 0.$$

Evaluating (10.7) on $(sA)^{\otimes n+1}$, we find

$$b_{C,1} \circ D_{n+1} + D_{n+1} \circ \left(\sum_{p+r=n} \mathrm{id}^{\otimes p} \otimes b_{A,1} \otimes \mathrm{id}^{\otimes r} \right) = 0,$$

or succinctly

$$(10.8) \quad [b_1, D_{n+1}] = 0.$$

Here $[b_1, -]$ is our notation for the differential on $\mathrm{Hom}_k((sA)^{\otimes n+1}, sC)$ induced by $b_{A,1}$ and $b_{C,1}$.

We now evaluate (10.7) on $(sA)^{\otimes n+2}$. We get

$$\begin{aligned} & b_{C,1} \circ D_{n+2} + b_{C,2} \circ (D_{n+1} \otimes \psi_1 + \psi_1 \otimes D_{n+1}) \\ & + D_{n+2} \circ \left(\sum_{p+r=n+1} \mathrm{id}^{\otimes p} \otimes b_{A,1} \otimes \mathrm{id}^{\otimes r} \right) + D_{n+1} \circ \left(\sum_{p+r=n} \mathrm{id}^{\otimes p} \otimes b_{A,2} \otimes \mathrm{id}^{\otimes r} \right) = 0 \end{aligned}$$

Written more nicely:

$$[b_1, D_{n+2}] + b_{C,2} \circ (D_{n+1} \otimes \psi_1) + \sum_{p+r=n} D_{n+1} \circ (\text{id}^{\otimes p} \otimes b_{A,2} \otimes \text{id}^{\otimes r}) + b_{C,2} \circ (\psi_1 \otimes D_{n+1}) = 0.$$

This may be rewritten as

$$(10.9) \quad 0 = d_{\text{Hoch}}(\bar{D}_{n+1})$$

where \bar{D}_{n+1} is the image of D_{n+1} in $H^1(\text{Hom}_k((sA)^{\otimes n+1}, sC)) = \text{Hom}_k(H^*(A)^{\otimes n+1}, H^*(C))_{-n+1}$, and where d_{Hoch} represents the Hochschild differential. Thus \bar{D}_{n+1} represents an element of $\text{HH}^{n+1}(H^*(A), H^*(C))_{-n+1}$.

At this point we use the idea that we may modify ψ_n as long as condition (10.5) remains valid. Let $\psi'_{\leq n}$ be like $\psi_{\leq n}$ except that ψ_n is replaced by $\psi'_n = \psi_n + \delta_n$, with $\delta_n : (sA)^{\otimes n} \rightarrow sC$. Then condition (10.5) remains valid for $\psi'_{\leq n}$ provided $[b_1, \delta_n] = 0$. We will assume this. Now let D' be like D but computed from $\psi'_{\leq n}$. Then we find by (10.6)

$$D'_{n+1} = D_{n+1} + b_{C,2} \circ (\delta_n \otimes \psi_1 + \psi_1 \otimes \delta_n) - \delta_n \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{A,2} \otimes \text{id}^{\otimes r} \right).$$

In other words

$$\bar{D}'_{n+1} = \bar{D}_{n+1} \pm d_{\text{Hoch}}(\bar{\delta}_n).$$

Combining this with (10.9), together with the assumption $\text{HH}^{n+1}(H^*(A), H^*(C))_{-n+1} = 0$ (as $n \geq 2$), it follows that we may modify ψ_n in such a way that $\bar{D}_{n+1} = 0$.

Let ψ_{n+1} be arbitrary. The condition $b_C \circ \psi_{\leq n+1} = \psi_{\leq n+1} \circ b_A$ when restricted to $(sA)^{n+1}$ may be succinctly written as

$$(10.10) \quad [b_1, \psi_{n+1}] = -(b_C \circ \psi_{\leq n} - \psi_{\leq n} \circ b_A) \mid (sA)^{\otimes n+1},$$

which may again be rewritten as

$$[b_1, \psi_{n+1}] = -D_{n+1}.$$

Since $\bar{D}_{n+1} = 0$, this equation has a solution. \square

Proof of Proposition 10.1.1. We may assume that M is a fibrant object in $C(\mathcal{C})$ for the standard model structure [3]. Put $A = \underline{\text{Hom}}_{\mathcal{C}}(M, M)$. The $H^*(B)$ -action on M is represented by a graded map $H^*(B) \rightarrow H^*(A)$. We may lift this map to a graded linear map $\phi : B \rightarrow A$, commuting with the differentials on B and A .

Since $H^*(A) = \text{Ext}_{\mathcal{C}}^*(M, M)$, the hypotheses together with Lemma 10.3.1 imply that ϕ may be lifted to an A_{∞} -morphism $\psi : B \rightarrow A$ such that $\psi_1 = \phi$. Then M becomes a homologically unital A_{∞} - B -module, that is, an object in $D_{\infty}(B, \mathcal{C})$. The proposition now follows by invoking Lemma 10.2.1, together with the commutative diagram

$$\begin{array}{ccc} D(B, \mathcal{C}) & \longrightarrow & D_{\infty}(B, \mathcal{C}) \\ \downarrow & & \downarrow \\ D(\mathcal{C})_{H^*(B)} & \xlongequal{\quad} & D(\mathcal{C})_{H^*(B)} \end{array}$$

\square

10.4. Proof of Proposition 10.1.2. Let $M, N \in D(B, \mathcal{C})$ be as in the statement of Proposition 10.1.2 and assume that N is fibrant for the model structure on $C(B, \mathcal{C})$ [25, Prop. 5.1]. Then it is easy to see that N is also fibrant when considered as an element of $C(\mathcal{C})$. In particular, an element $\varphi \in \text{Hom}_{D(\mathcal{C})^{H^*(B)}}(FM, FN)$ may be considered as an actual map $\varphi : M \rightarrow N$ in $C(\mathcal{C})$ commuting with the $H^*(B)$ -action, up to homotopy. We will construct a morphism $f : M \rightarrow N$ in $C_\infty(B, \mathcal{C})$ such that $f_1 = \varphi$. This is sufficient by Lemma 10.2.1.

Consider

$$b_{N,2} \circ (\text{id}_B \otimes f_1) - f_1 \otimes b_{M,2}.$$

This is a map $B \otimes_k M \rightarrow N$, which we may consider as a map $B \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$. The latter is zero on cohomology. Hence on the level of complexes of k -vector spaces it is zero up to homotopy. Call this homotopy $-f_2 : B \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$ and view it as a map $B \otimes_k M \rightarrow N$ in $C(\mathcal{C})$. Thus we have

$$(10.11) \quad b_{N,2} \circ (\text{id}_B \otimes f_1) - f_1 \otimes b_{M,2} = -b_{N,1} \circ f_2 + f_2 \circ (\text{id}_B \otimes b_{M,1} + b_{B,1} \otimes \text{id}_M).$$

Assume that we have constructed f_1, \dots, f_n . Define $f_{\leq n}$ as the comodule map $\mathbb{B}M \rightarrow \mathbb{B}N$ given by the Taylor coefficients $(f_1, \dots, f_n, 0, \dots)$. Assume furthermore that

$$(10.12) \quad b_N \circ f_{\leq n} = f_{\leq n} \circ b_M \quad \text{restricted to } (sB)^{\otimes i} \otimes M \text{ for } 0 \leq i \leq n-1.$$

It follows from (10.11) that we have already achieved this for $n \leq 2$. Our aim is now to construct f_{n+1} such that $b_N \circ f_{\leq n+1} = f_{\leq n+1} \circ b_M$ when restricted to $(sB)^{\otimes i} \otimes M$ for $0 \leq i \leq n$. As in the proof of Proposition 10.1.1, this will involve retroactively changing f_n .

Define $D = b_N \circ f_{\leq n} - f_{\leq n} \circ b_M$. Then we have $D_m = 0$ for $m = 0, \dots, n$. We will now show that $[b_1, D_{n+1}] = 0$. To do this, notice that

$$b_N \circ D + D \circ b_M = 0.$$

Evaluate this equation on $(sB)^{\otimes n} \otimes M$ and get

$$(10.13) \quad b_{N,1} \circ D_{n+1} + D_{n+1} \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \otimes \text{id}_M \right) + D_{n+1} \circ (\text{id}_B^{\otimes n} \otimes b_{M,1}) = 0$$

which is precisely the statement that $b_{N,1} \circ D_{n+1} + D \circ b_{M,1} = 0$.

We now want to take the adjoint map $(sB)^{\otimes n} \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$. To do this, first define

$$\mathbf{b}_{N,1} : \underline{\text{Hom}}_{\mathcal{C}}(M, N) \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N), \quad \mathbf{b}_{N,1}(f) = b_{N,1} \circ f$$

$$\mathbf{b}_{M,1} : \underline{\text{Hom}}_{\mathcal{C}}(M, N) \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N), \quad \mathbf{b}_{M,1}(f) = (-1)^{|f|} f \circ b_{M,1}$$

$$\mathbf{D}_{n+1} : (sB)^{\otimes n} \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N), \quad \mathbf{D}_{n+1}(sa_1, \dots, sa_n)(-) = D_{n+1}(sa_1, \dots, sa_n, -)$$

Later we will also use

$$\mathbf{b}_{N,2} : \underline{\text{Hom}}_{\mathcal{C}}(M, N) \otimes sB \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N), \quad (\mathbf{b}_{N,2}(f, sa))(m) = (-1)^{|sa| \cdot |f|} b_{N,2}(sa, f(m))$$

$$\mathbf{b}_{M,2} : \underline{\text{Hom}}_{\mathcal{C}}(M, N) \otimes sB \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N), \quad (\mathbf{b}_{M,2}(f, sa))(m) = (-1)^{|f|} (f \circ b_{M,2})(sa, m)$$

Now that we have all of these maps in place, let us go back to (10.13) and write down the corresponding equation for the adjoint map $(sB)^{\otimes n} \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$,

$$\mathbf{b}_{N,1} \circ \mathbf{D}_{n+1} + (-1)^{|D_{n+1}|} \mathbf{b}_{M,1} \circ \mathbf{D}_{n+1} + \mathbf{D}_{n+1} \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \right) = 0$$

which, remembering that D has degree 1, becomes

$$\mathbf{b}_{N,1} \circ \mathbf{D}_{n+1} - \mathbf{b}_{M,1} \circ \mathbf{D}_{n+1} + \mathbf{D}_{n+1} \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \right) = 0.$$

We may consider \mathbf{D}_{n+1} as an element in $\underline{\text{Hom}}_k((sB)^{\otimes n}, \underline{\text{Hom}}_{\mathcal{C}}(M, N))$. The induced differential on $\underline{\text{Hom}}_{\mathcal{C}}(M, N)$ is $b_{\underline{\text{Hom}}_{\mathcal{C}}(M, N), 1} = \mathbf{b}_{N,1} - \mathbf{b}_{M,1}$. Then $\underline{\text{Hom}}_k((sB)^{\otimes n}, \underline{\text{Hom}}_{\mathcal{C}}(M, N))$ is a complex with differential $[b_1, -]$. By the computation above, $[b_1, \mathbf{D}_{n+1}] = 0$. Define $\bar{\mathbf{D}}_{n+1}$ as the image of \mathbf{D}_{n+1} in

$$H^1(\underline{\text{Hom}}_k((sB)^{\otimes n}, \underline{\text{Hom}}_{\mathcal{C}}(M, N))) = \underline{\text{Hom}}_k(H^*(B)^{\otimes n}, \text{Ext}_{\mathcal{C}}^*(M, N))_{-n+1}.$$

Now evaluate $b_N \circ D + D \circ b_M = 0$ on $(sB)^{\otimes n+1} \otimes M$ to get

$$\begin{aligned} & b_{N,1} \circ D_{n+2} + b_{N,2} \circ (\text{id} \otimes D_{n+1}) + \\ & + D_{n+2} \circ \left(\sum_{p+r+2=n+2} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \otimes \text{id}_M + \text{id}_B^{\otimes n+1} \otimes b_{M,1} \right) \\ & + D_{n+1} \circ \left(\sum_{p+r+2=n+1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \otimes \text{id}_M + \text{id}_B^{\otimes n} \otimes b_{M,2} \right) = 0. \end{aligned}$$

Rewrite the sums as

$$\begin{aligned} & b_{N,1} \circ D_{n+2} + D_{n+2} \circ (\text{id}_B^{\otimes n+1} \otimes b_{M,1}) \\ & + D_{n+2} \circ \left(\sum_{p+r=n} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \otimes \text{id}_M \right) \\ & + b_{N,2} \circ (\text{id} \otimes D_{n+1}) + D_{n+1} \circ (\text{id}_B^{\otimes n} \otimes b_{M,2}) \\ & + D_{n+1} \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \otimes \text{id}_M \right) = 0 \end{aligned}$$

By adjointness this gives maps $(sB)^{\otimes n+1} \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$ such that (remember that D has degree 1, so $(-1)^{|D_{n+2}|} = -1$)

$$\begin{aligned} & \mathbf{b}_{N,1} \circ \mathbf{D}_{n+2} - \mathbf{b}_{M,1} \circ \mathbf{D}_{n+2} \\ & + \mathbf{D}_{n+2} \circ \left(\sum_{p+r=n} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \right) \\ & + \mathbf{b}_{N,2} \circ (\text{id} \otimes \mathbf{D}_{n+1}) - \mathbf{b}_{M,2} \circ (\text{id} \otimes \mathbf{D}_{n+1}) \\ & + \mathbf{D}_{n+1} \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \right) = 0; \end{aligned}$$

this means that $\bar{\mathbf{D}}_{n+1}$ is a cocycle in $(\underline{\text{Hom}}_k(H^*(B)^{\otimes n}, \text{Ext}_{\mathcal{C}}^*(M, N), d_{\text{Hoch}})_{-n+1}$, hence an element of

$$\text{HH}^n(H^*(B), \text{Ext}^*(M, N))_{-n+1}.$$

Now let $f'_{\leq n}$ be like $f_{\leq n}$ except that f_n is replaced by $f_n + \delta_n$, where $\delta_n : (sB)^{\otimes n-1} \otimes M \rightarrow N$ is such that $[b_1, \delta_n] = 0$, and let $D' = b_N \circ f'_{\leq n} - f'_{\leq n} \circ b_M$. Since $b_N \circ f'_{\leq n} - f'_{\leq n} \circ b_M|_{(sB)^{\otimes i} \otimes M} = 0$ for $i = 0, \dots, n-1$, we still have $D'_i = 0$ for $i = 1, \dots, n$, whereas

$$D'_{n+1} = D_{n+1} + b_{N,2} \circ (\text{id} \otimes \delta_n) - \delta_n \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} + \text{id}^{\otimes n-1} \otimes b_{M,2} \right)$$

The corresponding map $\mathbf{D}'_{n+1} : (sB)^{\otimes n} \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$ is given by

$$\mathbf{D}'_{n+1} = \mathbf{D}_{n+1} + \mathbf{b}_{N,2} \circ (\text{id} \otimes \delta_n) - \delta_n \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \right) - \mathbf{b}_{M,2} \circ (\text{id} \otimes \delta_n)$$

where $\delta_n : (sB)^{\otimes n-1} \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$, $\delta_n(sa_1, \dots, sa_{n-1})(-) = \delta_n(sa_1, \dots, sa_{n-1}, -)$. Hence $\bar{\mathbf{D}}'_{n+1} = \bar{\mathbf{D}}_{n+1} \pm d_{\text{Hoch}}(\delta_n)$. Since we have assumed $\text{HH}^n(H^*(B), \text{Ext}^*(M, N))_{-n+1} = 0$, it means $\bar{\mathbf{D}}_{n+1}$ is a coboundary, and hence we can assume it is actually zero after replacing it with $\bar{\mathbf{D}}_{n+1} \pm d_{\text{Hoch}}(\delta_n)$.

Given a map f_{n+1} , the condition that $f_{\leq n+1}$ needs to satisfy to complete the induction step is $b_N \circ f_{\leq n+1} = f_{\leq n+1} \circ b_M$ when restricted to $(sB)^{\otimes n} \otimes M$. This gives

$$[b_1, f_{n+1}] = -(b_N \circ f_{\leq n} - f_{\leq n} \circ b_M) \quad (\text{on } (sB)^{\otimes n} \otimes M)$$

which gives

$$[b_1, f_{n+1}] = -D_{n+1}$$

and since $\bar{\mathbf{D}}_{n+1} = 0$, and hence $\bar{D}_{n+1} = 0$, this equation has a solution. \square

10.5. Proof of Proposition 10.1.3. Let $M, N \in D(B, \mathcal{C})$ be as in the statement of Proposition 10.1.3 and assume that N is fibrant for the model structure on $C(B, \mathcal{C})$ [25, Prop. 5.1]. Assume that $g : M \rightarrow N$ is a morphism in $C_{\infty}(B, \mathcal{C})$ which is sent to zero in $\text{Hom}_{D(\mathcal{C})_{H^*(B)}}(FM, FN) \subset \text{Hom}_{D(\mathcal{C})}(M, N)$.

Assume that $g_i = 0$ holds for $i \leq n$. We will change g by a homotopy h , with $h_i = 0$ for $i \neq n, n+1$ such that $g_i = 0$ for $i \leq n+1$. Iterating this we find that our original $g : M \rightarrow N$ is homotopic to zero.

Consider first $n = 0$. Then, since g is zero in $\text{Hom}_{D(\mathcal{C})}(M, N)$, we have $h_1 : M \rightarrow N$ such that $g_1 = b_{N,1} \circ h_1 + h_1 \circ b_{M,1}$. Let $h : \mathbb{B}M \rightarrow \mathbb{B}N$ be the coalgebra map $(h_1, 0, \dots)$ and put $g' = g - (b_N \circ h + h \circ b_M)$. Then $g'_1 = 0$.

Now assume $n \geq 1$. When evaluating $b_N \circ g = g \circ b_M$ on $(sB)^{\otimes n} \otimes M$ we get

$$[b_1, g_{n+1}] = 0.$$

Hence g_{n+1} is a cocycle in the complex $\underline{\text{Hom}}_k((sB)^{\otimes n} \otimes M, N)$ with differential $[b_1, -]$. We may consider it as an element of $\underline{\text{Hom}}_k((sB)^{\otimes n}, \underline{\text{Hom}}_{\mathcal{C}}(M, N))_0$ by adjointness. Call the adjoint map \mathbf{g}_{n+1} , and $\bar{\mathbf{g}}_{n+1}$ its image in the cohomology $H^0(\underline{\text{Hom}}_k((sB)^{\otimes n} \otimes M, N)) = \underline{\text{Hom}}_k(H^*(B)^{\otimes n}, \text{Ext}^*_{\mathcal{C}}(M, N))_{-n}$.

Now evaluate $b_N \circ g = g \circ b_M$ on $(sB)^{\otimes n+1} \otimes M$ to get:

$$\begin{aligned} & b_{N,1} \circ g_{n+2} + b_{N,2} \circ (\text{id} \otimes g_{n+1}) + \\ & - g_{n+2} \circ \left(\sum_{p+r+2=n+2} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \otimes \text{id}_M + \text{id}_B^{\otimes n+1} \otimes b_{M,1} \right) \\ & - g_{n+1} \circ \left(\sum_{p+r+2=n+1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \otimes \text{id}_M + \text{id}_B^{\otimes n} \otimes b_{M,2} \right) = 0 \end{aligned}$$

By adjointness this gives maps $(sB)^{\otimes n+1} \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(M, N)$ such that

$$\begin{aligned} & \mathbf{b}_{N,1} \circ \mathbf{g}_{n+2} - \mathbf{b}_{M,1} \circ \mathbf{g}_{n+2} \\ & - \mathbf{g}_{n+2} \circ \left(\sum_{p+r=n} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \right) \\ & + \mathbf{b}_{N,2} \circ (\text{id} \otimes \mathbf{g}_{n+1}) - \mathbf{b}_{M,2} \circ (\text{id} \otimes \mathbf{g}_{n+1}) \\ & - \mathbf{g}_{n+1} \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \right) = 0 \end{aligned}$$

Since g has degree zero, this means that $d_{\text{Hoch}}(\bar{\mathbf{g}}_{n+1}) = 0$, i.e. that $\bar{\mathbf{g}}_{n+1}$ is a cocycle in $(\underline{\text{Hom}}_k(H^*(B)^{\otimes n}, \text{Ext}_{\mathcal{C}}^*(M, N), d_{\text{Hoch}})_{-n}$ hence an element of

$$\text{HH}^n(H^*(B), \text{Ext}_{\mathcal{C}}^*(M, N))_{-n};$$

which we have assumed to be zero for $n \geq 1$, hence $\bar{\mathbf{g}}_{n+1}$ is a coboundary. Which means that there exists a $\bar{\mathbf{h}}_n \in (\underline{\text{Hom}}_k(H^*(B)^{\otimes n-1}, \text{Ext}_{\mathcal{C}}^*(M, N))_{-n}$ such that $\bar{\mathbf{g}}_{n+1} = d_{\text{Hoch}}(\bar{\mathbf{h}}_n)$.

We may lift $\bar{\mathbf{h}}_n$ to a map $\mathbf{h}_n : (sB)^{\otimes n-1} \rightarrow \text{Hom}_{\mathcal{C}}(M, N)$ such that $[b_1, \mathbf{h}_n] = 0$, or equivalently by adjointness a map $h_n : (sB)^{\otimes n-1} \otimes M \rightarrow N$. Because $\bar{\mathbf{g}}_{n+1} = d_{\text{Hoch}}(\bar{\mathbf{h}}_n)$, their difference is a boundary:

$$(10.14) \quad \mathbf{g}_{n+1} - (\mathbf{b}_{N,2}(\text{id} \otimes \mathbf{h}_n) + \mathbf{b}_{M,2} \circ (\text{id} \otimes \mathbf{h}_n) - (-1)^{|\mathbf{h}|} \mathbf{h}_n \circ (\sum \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r})) = [b_1, \mathbf{h}_{n+1}]$$

for some h_{n+1} in $\underline{\text{Hom}}_k((sB)^{\otimes n}, \underline{\text{Hom}}_{\mathcal{C}}(M, N))_{-1}$.

Let h be the comodule map $\mathbb{B}M \rightarrow \mathbb{B}N$ such that $h_i = 0$ for $i \neq n, n+1$ and h_n, h_{n+1} as above. These h will be the required homotopy. In fact, by rewriting (10.14), we obtain

$$\begin{aligned} \mathbf{g}_{n+1} &= \mathbf{b}_{N,1} \circ \mathbf{h}_{n+1} - \mathbf{b}_{M,1} \circ \mathbf{h}_{n+1} \\ &+ \mathbf{h}_{n+1} \circ \left(\sum_{p+r=n} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \right) \\ &+ \mathbf{b}_{N,2} \circ (\text{id} \otimes \mathbf{h}_n) - \mathbf{b}_{M,2} \circ (\text{id} \otimes \mathbf{h}_n) \\ &+ \mathbf{h}_n \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \right) \end{aligned}$$

which, by adjointness, gives us (h has degree -1 so this accounts for the change in signs)

$$(10.15) \quad \begin{aligned} g_{n+1} &= b_{N,1} \circ h_{n+1} + b_{M,1} \circ h_{n+1} \\ &+ h_{n+1} \circ \left(\sum_{p+r=n} \text{id}^{\otimes p} \otimes b_{B,1} \otimes \text{id}^{\otimes r} \right) \\ &+ b_{N,2} \circ (\text{id} \otimes h_n) + b_{M,2} \circ (\text{id} \otimes h_n) \\ &+ h_n \circ \left(\sum_{p+r=n-1} \text{id}^{\otimes p} \otimes b_{B,2} \otimes \text{id}^{\otimes r} \right) \end{aligned}$$

Define $g' = g - (b_N \circ h + h \circ b_M)$. It follows from (10.15) that $g'_{n+1} = 0$ and hence we are done. \square

APPENDIX A. PROOF OF PROPOSITION 7.1 FOR THE TWO LOOP QUIVER

Let Q be the two loop quiver, and let $\alpha \in \mathbb{N} - \{0\}$. The proof of Proposition 7.1 depends crucially on the construction of a representation U over LQ for L/k a field of transcendence degree three such that $\mathrm{HH}^3(L, \mathrm{End}_{kQ}(U)) \neq 0$.

Assume we try to find our U as defined over the generic point of a closed subvariety of dimension three in a suitable $M_{\alpha, \lambda, Q}$. In this case there is only one possibility for λ , namely $\lambda = 0$. If $\alpha = 1$ then $\dim M_{\alpha, 0, Q} = 2$ which is too small for our purposes. However, when $\alpha > 1$ then α is divisible and so Proposition 3.5.2 does not apply.

We proceed as follows. Assume $n = \alpha > 1$ and put $A = kQ$. Then it is well known [32] that $U_{n, A}$ is nonempty and that \mathcal{A}_n is not split. In fact, it is generically a division algebra⁶. Let x be the generic point of a three dimensional irreducible subvariety of $U_{n, A}$ and put $K = k(x)$. Let $C = i_x^* \mathcal{A}_n$. Then C is a central simple algebra of rank n^2 over K . Thus we have $C = M_m(D)$ where D is a division algebra such that $[D : K] = p^2$ with $n = pm$. Let L/K be a maximal subfield of D . Then $L \otimes_K C = M_m(L \otimes_K D) = M_{mp}(L) = \mathrm{End}_L(U)$ where $U = L^n$. As in the proof of Lemma 3.2.1 we obtain

$$\mathrm{End}_A(U) = \mathrm{End}_C(U)$$

The following lemma does what we want by the Hochschild-Kostant-Rosenberg theorem [17].

Lemma A.1. *One has $\mathrm{HH}^*(L, \mathrm{End}_C(U)) \cong \mathrm{HH}^*(K, L)$.*

Proof. By Morita theory we have

$$\mathrm{End}_C(U) = \mathrm{End}_D(U_0)$$

where $U_0 = L^p$. So we may and we will assume that $C = D$, $U = U_0$, $m = 1$, $n = p$.

Since D/K is central simple we have an isomorphism of algebras

$$D \otimes_K D^\circ \rightarrow \mathrm{End}_K(D) : d \otimes d' \mapsto (x \mapsto dx d')$$

Taking centralisers of $1 \otimes_K L$ and $L \otimes_K L$ on both sides, we find corresponding isomorphisms

$$(A.1) \quad D \otimes_K L \cong \mathrm{End}_L(D_L)$$

$$(A.2) \quad L \otimes_K L \cong \mathrm{End}_{L \otimes_K L}({}_L D_L),$$

where $D_L, {}_L D_L$ denote D viewed respectively as a right L -module and as a left-right L -bimodule. Since $L \otimes_K L$ is a direct sum of fields, (A.2) implies that ${}_L D_L$ is isomorphic to $L \otimes_K L$ as L -bimodules. Since we also have $L = \mathrm{Hom}_K(L, K)$ as L -vector spaces (both are one-dimensional), we obtain that ${}_L D_L$ is also isomorphic to

$$(A.3) \quad {}_L D_L \cong \mathrm{Hom}_K(L, K) \otimes_K L = \mathrm{Hom}_K(L, L)$$

as L -bimodules.

The equality (A.1) implies that $U \cong D_L$ with the standard left D -module structure on D . Thus

$$\mathrm{End}_D(U) = \mathrm{End}_D(D_L) = D^\circ$$

⁶Suitably called the generic division algebra of index n .

with the L -bimodule structure on D° given by the left and right action. Combining this with (A.3) we get

$$\begin{aligned}\mathrm{HH}^*(L, \mathrm{End}_C(U)) &\cong \mathrm{HH}^*(L, \mathrm{Hom}_K(L, L)) \\ &\cong \mathrm{HH}^*(K, L)\end{aligned}$$

where in the last line we have used Corollary 5.5. \square

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